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Flow and stability studies in porous media based on some non-Darcian models.

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FLOW AND STABILITY STUDIES IN POROUS MEDIA BASED ON SOME NON-DARCIAN MODELS

by

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ABSTRACT

This thesis deals with the study of certain flow and stability problems in porous media using some non-Darcian models. It is divided into three parts, which are essentially independent of each other.

The first part deals with the theoretical study related with the solution of the Brinkman equation. An exact Cartesian-tensor form solution of the Brinkman equation is found, which facilitates the study of various boundary value problems. The problems of the flow past a porous sphere and creeping flow past a porous spherical shell are considered to illustrate the point. Then, a theoretical look is taken at the problem of steady convection in porous media, again using the generalized Brinkman equation. A variational formulation is introduced to define a weak solution and existence, uniqueness, and regularity of the weak solution are discussed.

In the second part, two stability problems in porous media are examined. First, a theoretical investigation for the onset of Rayleigh-Bénard convection in a porous layer, using the Brinkman equation with anisotropic permeability, is presented. The critical Rayleigh numbers, within the framework of linear theory, for both rigid and free boundaries, are calculated. The effect of considering anisotropy in the Brinkman equation is brought out. Next, an energy method is used to study the nonlinear stability of a rotating porous layer. Here the Brinkman-Boussinesq model is employed and a generalized energy functional is used to determine the energy stability bound. A comparison is also made with the linear instability bound.

In the third part, another non-Darcian model is used to study the problem of unsteady flow of a power-law fluid in a porous medium. A mathematical analysis for two typical initial-boundary value problems, which correspond to well-test cases in the oil industry, is presented.

To my wife Yin and my son Yiding, without their understanding, encouragement and love, this dissertation would not have been possible.

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Chapter 1. Introduction

Flow through porous media has attracted considerable research activity in recent years because of worldwide concern with issues such as energy self-sufficiency and pollution of the environment. Areas of applications include the insulation of buildings and equipment, energy storage and recovery, geothermal reservoirs, nuclear waste disposal, chemical reactor engineering, and the storage of ore. Geophysical applications range from the flow of groundwater around hot intrusions to the stability of snow against avalanches.

A porous medium generally is an extremely complicated network of channels and obstructions. In order to study the flow of fluids through porous media, it is necessary to clarify what is understood by the properties of porous media and fluids. Below we only give some physical definitions pertinent to our study and then move on to a mathematical description.

In general terms a porous medium is a solid body containing voids or pores that are interconnected or unconnected and are dispersed randomly or in an ordered geometry. Porosity is defined as the ratio of pore volume to total volume of the porous material. It has no dimensions. Permeability of porous media is defined as the ability to let fluid flow through its interconnected pore network. It is a measure of fluid conductivity. Its dimension is a length squared. Tortuosity is the relative average length of the flow path of a fluid particle from one side of the porous medium to the other. It is a dimensionless quantity.

Historically, the study of the flow of viscous fluids through permeable materials began with Darcy's law (1856). In modern notation this is expressed by

$$u = -\frac{k}{\mu}\frac{\partial p}{\partial x}, \tag{1-1}$$

where k is the permeability of the porous medium, μ is the dynamic viscosity of

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX the fluid, u is the average velocity, p is the pressure.

In three dimensions, equation (1-1) generalizes to

$$\mathbf{u} = -\mu^{-1}\mathbf{k} \cdot \nabla p, \tag{1-2}$$

where now the permeability $\mathbf{k}$ is in general a second-order tensor, for the case of an isotropic medium the permeability is a scalar k; and $\mathbf{u} = (u, v, w)$ denotes the volume average macroscopic velocity of the fluid over a representative elementary volume. This quantity has been given various names by different authors, such as seepage velocity, filtration velocity, and volumetric flux density.

Darcy's law has been verified by the results of many experiments. Theoretical derivations for it have been obtained by many researchers. Since it is not possible to know what is happening in each of the many pores, the best one can expect is to obtain a knowledge of an average of physical quantities over volumes of regions containing many pores. There are several derivations of Darcy's law. The usual procedure of deriving the law governing the macroscopic variables is to begin with the Navier-Stokes equations to obtain the macroscopic equations by averaging. Thus Hubbert (1956) and Hall (1956) derived Darcy's law by integrating the Navier-Stokes equations over a representative elementary volume of a porous medium. Poreh and Elata (1966), and Whitaker (1966) followed Hubbert and Hall and used an averaging process to derive Darcy's law in a rigorous fashion. Neuman (1977) derived Darcy's law for anisotropic porous media by using a formal averaging procedure, and Whitaker (1986) showed that the same law was true for a nonhomogeneous and non-periodic porous medium. Rubinstein and Torquato (1989) used an ensemble-average approach to derive Darcy's law while Tartar (1980) and Keller (1980) derived it by giving more rigorous mathematical proofs which employed the homogenization method for the case of periodic geometry.

Darcy's law has been applied to a vast array of problems involving flow through porous media and has proven to be a reliable model for flow in an infinite porous material or in the interior of finite porous materials. Though it is successful for interior flows, Darcy's law does have some limitations. This model entirely neglects the effects of a solid boundary and the inertial forces on fluid flow through porous media. These effects are expected to become more significant near the boundary and in high porosity media, thus causing the application of Darcy's law to be inappropriate (Palm *et al.* 1972). Mathematically, since the order of Darcy's law is lower than the Navier-Stokes equations, the no-slip boundary condition cannot be imposed on the impermeable boundary. When the flow past a porous body of finite size is considered, the interior flow must be matched with the exterior pure fluid flow at the boundary surface. This is not possible owing to the reduced order of Darcy's law. To overcome this difficulty, Beavers and Joseph (1967) suggested that the law be retained for the interior flow but that to match the exterior pure fluid flow, the boundary condition be modified by the form

$$\frac{\partial u}{\partial y}\Big|_{y=0^+} = \frac{\alpha}{k^{1/2}}(u - u_D)\,. \tag{1-3}$$

Here u is the local average tangential velocity outside the body, u_D is the tangential velocity given by Darcy's law and α is the slip coefficient, a dimensionless constant depending on the local geometry of the interstices, that is, depending on the closed spacing of pores. This slip condition was proposed on heuristic grounds. Saffman (1971) gave theoretical justification for it and showed that the condition could be derived in the form

$$u = \frac{k^{1/2}}{\alpha}\frac{\partial u}{\partial y}\Big|_{y=0^+} + O(k)\,. \tag{1-4}$$

With the aid of condition (1-3) or (1-4), Darcy's law is applicable to the problems of a porous medium adjacent to a viscous fluid. There are still some unclear points concerning this condition as shown by Larson and Higdon (1986). Their calculations indicate that the definition of a slip coefficient for a porous medium is meaningful only for extremely dilute systems. In particular, these authors found this approach fundamentally flawed in the sense that microscopic changes in the position of the nominal interface, too small to be discernable on a macroscopic level, lead to $O(1)$ changes in the slip coefficient which is thus not defined in a self-consistent fashion. That is, it is not possible to define a consistent value for slip coefficient for any medium.

Due to the limitations of Darcy's law in some circumstances, the interest in non-Darcy effects has been increasing in recent years. In order to overcome the matching of interior and exterior flow, an alternate to Darcy's law has been proposed. This is the Brinkman equation

$$-\nabla p + \mu^* \nabla^2 \mathbf{u} - \frac{\mu}{k}\mathbf{u} = 0\,, \tag{1-5}$$

which is of the same order as the Stokes' equation. The first two terms represent the divergence of the local average viscous stress tensor incorporating an effective viscosity μ^*, while the third term represents the distributed resistance of the solid inclusion. Brinkman (1947) assumed $\mu^* = \mu$ and calculated the force on the surface of the sphere based upon the above equation. On equating the total force on the sphere contained in a column of the media to the Darcy's drag on the column he concluded that the porosity ϕ must be such that $\frac{1}{3} < \phi < 1$. Lundgren (1972) re-examined this problem and concluded that in order for the Brinkman equation to be applicable, one should have $0.6 < \phi < 1$. Though Brinkman's derivation was heuristic, subsequent investigators have rigorously established the validity of this equation for low volume fraction of solids. In the literature, one can find such derivation from Tam (1969), Slattery (1969), Childress (1972), Lundgren (1972), Howells (1974), Hinch (1977), Rubinstein (1986). The experimental checks can be seen from Matsumoto and Suganuma (1977).

Darcy's equation (1-2) holds when the seepage velocity u is sufficiently small. If the seepage velocity is increased, the inertial terms become appreciable and deviations from Darcy's law are observed. Forchheimer (1901) recognized this breakdown and suggested the addition of a quadratic term to Darcy's equation. A modern form of this equation is

$$\nabla p = -\frac{\mu}{k}\mathbf{u} - \frac{C_f}{\sqrt{k}}\rho_f|\mathbf{u}|\mathbf{u}, \tag{1-6}$$

where C_f is a dimensionless form drag coefficient which varies with the nature of the porous medium and ρ_f is density of the fluid. It has been suggested that this term arises because of averaging of microscopic drag forces and not by averaging microscopic inertial terms.

Some authors, Vafai and Tien (1981) and Ettefagh *et al.* (1991), combine the Brinkman and Forchheimer expressions to form a semi-empirical equation

$$\frac{\rho_f}{\phi}\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\phi}\mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu^*\nabla^2\mathbf{u} + \rho_f\mathbf{f} - \frac{\mu}{k}\mathbf{u} - \frac{\rho_f C_f}{\sqrt{k}}|\mathbf{u}|\mathbf{u}, \tag{1-7}$$

where $\mathbf{f}$ is body force. The steady form of this equation seems to have been first suggested by Vafai and Tien (1981). The above form or variant of it is in much use in engineering literature.

Finally we mention another non-Darcy model which has been used to model the non-Newtonian flow of a power-law fluid in porous media. In place of the Brinkman term, in a two dimensional motion, the following equation is used (Parnas and Cohen 1987)

$$\frac{dp}{dx} + \frac{\mu}{k}u + \frac{d\tau}{dy} = 0, \tag{1-8}$$

where

$$\tau = -K\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy},$$

and K is a power law parameter. For $n = 1$ the above reduces to Brinkman's equation for the flow of a Newtonian fluid through porous media.

We now return to Saffman's assessment of the Brinkman equation. Saffman (1971) examined the applicability of Brinkman's equation to the flow near the surface of a porous domain using averaged equations and gave a more general result

$$\mu \int R_{ij}(x-\xi)u_j(\xi)\, d\xi = \mu\nabla^2 u_i - \frac{\partial p}{\partial x_i}. \tag{1-9}$$

The integral on the left in (1-9) represents the distributed resistance of the averaged domain, evaluated at the point x. Each point ξ contributes a resistance kernel depending on both ξ and x. Brinkman's equation results when the distributed resistance is obtained by approximating the resistance kernel with

$$R_{ij}(x-\xi) = \frac{\delta_{ij}}{k}\delta(x-\xi).$$

The corresponding resistance term for an anisotropic medium is

$$R_{ij}(x-\xi) = M_{ij}\delta(x-\xi), \tag{1-10}$$

and the resulting Brinkman equation is

$$-\nabla p + \mu\nabla^2\mathbf{u} - \mu\mathbf{M}\mathbf{u} = 0, \tag{1-11}$$

where $\mathbf{M}$ is a constant matrix. The simple delta-function resistance kernels hold strictly for points in the interior of the porous medium, well away from any surface. More generally, we may expand the kernel as a series in the delta function and its derivatives, which would through integration lead to terms involving derivatives of $\mathbf{u}$ and resistance terms which vary with x across the porous boundary layer.

It should be pointed out that the Brinkman correction term might be very small in the interior region of a porous material. This can be estimated if a scaling analysis is imposed. Let L be the characteristic length scale for volume average quantities and l be the characteristic pore diameter such that $l << L$. We have the estimates

$$\begin{aligned} \mu^* \nabla^2 \mathbf{u} &= O(\mu \mathbf{u}/L^2) \\ \frac{\mu}{k}\mathbf{u} &= O(\mu \mathbf{u}/l^2) \end{aligned} \quad . \tag{1-12}$$

However, it can be argued that as an applicable approximation, Brinkman's model is one of the most important equations to describe flow in porous media. There is no doubt that use of Brinkman's model becomes more significant and provides satisficatory results for flow in high porosity media and near the boundary. In comparison with Darcy's model, the conclusion as to which is a superior model will likely depend upon the type of porous material and upon the nature of problems to be investigated. Moreover, the recent upsurge of utilizing high porosity media in contemporary technology provides further impetus for obtaining thorough understanding of the boundary and inertial effects. There are also some situations where one wishes to compare flows in porous media with those in clear fluids. The use of Brinkman's equation is very convenient to link both extreme limits when the parameter k, appearing in this equation, tends to zero or to infinity. Because of all the above reasons, the research interest by using the Brinkman model has increased in recent years. There are a number of studies and investigations based on it in the literature. Some of the important ones are cited here: Katto and Masuoka (1967), Walker and Homsy (1977), Vafai and Tien (1981), Rudraiah and Musuoka (1982), Hsu and Cheng (1985), Kaviany (1987), Beckermann and Viskanta (1987), Cheng(1987), Vafai and Kim (1990), Lauriant and Vafai (1991) and Chen and Chen (1992). We remark that in comparison to the papers based on Darcy's law, works using the non-Darcy models are very few.

For the reasons stated above, it seems worthwhile to look at more problems based on the use of the Brinkman model. In view of the nature of the problems studied, the presentation here is mainly divided into three parts: The first part deals with the structure of the basic solutions of the Brinkman equation. The second part is concerned with some stability properties of the solutions associated with the generalized Brinkman equation. The third part deals with the study of unsteady flow problems of a power law fluid in porous media.

In Chapter 2, a certain form of the exact solution for the Brinkman equation is found. A feature of the method used here is that we develop the solution in the form of Cartesian tensors. The solutions obtained in this way will be directly applicable to porous-flow problems for which the boundary conditions are given in the Cartesian tensor form and to those that can be easily written in this manner. Several applications, such as finding the drag of a porous sphere in fluid flow and creeping flow past a porous spherical shell are considered to illustrate the simplicity and applicability of the method. It is believed that studies of the porous shell problem are useful in floc sedimentation problems. In the last part of this chapter we also discuss the change in boundary conditions if Darcy's law were to be used within the porous shell.

Chapter 3 takes a theoretical look at the problem of steady convection in a porous medium based on the generalized Brinkman-Oberbeck-Boussinesq equation. A variational formulation is introduced to define a class of weak solutions. The basic concerns such as existence, uniqueness and regularity of weak solutions are discussed. The addition of the Laplacian term to the velocity field in Darcy's law allows us to use Galerkin's method in a Hilbert space to prove the existence of the solution, and to use some deep results of elliptical partial differential equations to study other properties of the solutions. We find certain restrictions on the Rayleigh number in order for the solution of the system of equations to be unique.

In Chapter 4, a theoretical investigation for the onset of Rayleigh-Bénard convection in a porous layer, using both the Brinkman equation and anisotropic permeability, is presented. This seems to be the first study which considers anisotropic effects for convective instability in Brinkman's equation. The critical Rayleigh numbers at the marginal stability within the framework of linear stability theory are calculated for both free and rigid boundaries. Our analysis is likely to be useful for moderate porosity materials. Comparison is made with the limiting cases for the low porosity Darcy approximation as well as for pure viscous fluid. In the course of our discussion we also find the range of applicability of the Brinkman, Darcy and pure viscous models. In this regard, we put Nield's (1983) concern in proper perspective.

In Chapter 5, a nonlinear stability analysis is applied to the porous layer with rotation based on the Darcy-Brinkman-Boussinesq model. The energy method, a well developed new theory, is adopted to investigate the problem. The new energy functional variables, chosen as Liapunov function, require including the terms which are the derivatives of velocity and temperature. This makes it impossible for one to use Darcy's equation in the rotating porous Bénard problem, as Darcy's law is of lower order as compared to the Navier-Stokes equations. In this study the rigorous analysis gives a significant energy stability bound which takes the rotation effect into account and reflects the effect of Darcy's number in porous media. The linear instability bound is also briefly analyzed and compared to the results obtained from the energy stability theory.

In Chapter 6, a study of two typical initial-boundary value problems of the flow of a power law fluid in porous media, is presented. The problems considered belong to well-test cases in the oil industry. The existence of the solution of a nonlinear degenerate parabolic equation is gained as the limit of a sequence of classical solutions to the approximate non-degenerate equations with the initial-boundary value conditions. A decisive step in the above process is to derive a Hölder estimation of the approximate solutions. The uniqueness and regularity of the solution are also discussed.

It should be remarked that while the work in this thesis is mostly theoretical, some numerical simulation work in porous media has been carried out by Hamdan (1989). Some useful information about flow problems in porous media using non-Darcian models is also given by Rudraiah *et al.* (1979).

For later purposes, some notations and basic definitions will be given in the rest of this chapter.

Let $\mathcal{D}(\Omega)$ be the space of C^∞ functions with compact support contained in a domain Ω in R^n ($n = 2, 3$ for our study). A space $\mathcal{V}$ (without topology) is defined by

$$\mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega) \mid div\mathbf{u} = 0\}. \tag{1-13}$$

As usual, the scalar products and norms in Hilbert space $L^2(\Omega)$, $H^m(\Omega)$ are, respectively, denoted by

$$\begin{gathered}(u,v) = \int_\Omega uv\,dx, \qquad \|u\|_{L^2(\Omega)} = (u,u)^{1/2},\\ ((u,v))_{H^m(\Omega)} = \sum_{|j|\le m}(D^j u, D^j v), \qquad \|u\|_{H^m(\Omega)} = ((u,u))^{1/2},\\ D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1}\dots\partial x_n^{j_n}}, \qquad |j| = j_1 + \cdots + j_n,\end{gathered} \tag{1-14}$$

and the norms in Banach space $L^p(\Omega)$, $W^{m,p}(\Omega)$ are defined by

$$\begin{gathered}\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p\,dx\right)^{1/p},\\ \|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|j|\le m}\|D^j u\|^p_{L^p(\Omega)}\right)^{1/p}.\end{gathered} \tag{1-15}$$

In the Hilbert space $H_0^1(\Omega)$ an equivalent norm is given by

$$\|u\|_{H_0^1(\Omega)} = \left(\sum_{i=1}^{n} \|D_i u\|_{L^2(\Omega)}^2 \right)^{1/2}, \qquad D_i = \frac{\partial}{\partial x_i}. \tag{1-16}$$

We shall use the same notations for the scalar products and norms in product spaces $\mathbf{L}^2(\Omega) = (L^2(\Omega))^n$, $\mathbf{H}^m(\Omega) = (H^m(\Omega))^n$ and $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^n$.

In this thesis, both energy and linear methods are used to treat stability problems. In the energy stability method, an energy function (or a generalized functional) is chosen to establish a differential inequality. Then, the sufficient conditions are found to ensure stability of the problem. In accordance with standard literature on stability theory, some pertinent definitions follow (cf. Straughan 1992 p. 61).

Let H be a Hilbert space endowed with a scalar product $(\ ,\ )$ and associated norm $\|\cdot\|$. We consider in H the following initial value problem

$$u_t + Lu + N(u) = 0, \qquad u(0) = u_0. \tag{1-17}$$

Here L represents a linear operator (possibly unbounded), and N is a nonlinear operator with $N(0) = 0$ in order that (1-17) admits the null solution. We assume:

(i) L is a densely defined closed operator such that $(L - \lambda I)^{-1}$ is compact for some complex number λ (I is the identity operator in H), that is, L is an operator with compact resolvent.

(ii) The bilinear form associated with L is defined (and bounded) on a space H^*, which is compactly embedded in H.

(iii) The nonlinear operator N satisfies the condition

$$(N(u), u) \geq 0, \qquad \forall u \in D(N),$$

where $D(\cdot)$ denotes the domain of the associated operator.

Because of to (i), the following result is true (Kato 1976 p. 185-187). The spectrum of the operator L consists entirely of an at most denumerable number of eigenvalues $\{\sigma_n\}_{n\in \mathbf{N}}$ with finite (both algebraic and geometric) multiplicities and, moreover, such eigenvalues can cluster only at infinity.

Since the operator L is in general non-symmetric the eigenvalues, which satisfy the equation

$$L\Phi = \lambda\Phi,$$

are not necessarily real; they may, however, be ordered in the following manner

$$\Re(\sigma_1) < \Re(\sigma_2) < \cdots < \Re(\sigma_n) < \cdots . \tag{1-18}$$

Definition 1. The null solution of (1-17) is said to be linearly stable if and only if

$$\Re(\sigma_1) > 0. \tag{1-19}$$

Definition 2. The null solution of (1-17) is said to be nonlinearly stable if and only if for $\epsilon > 0$ there is a $\delta = \delta(\epsilon)$ such that

$$1) \qquad \|u_0\| < \delta \Rightarrow \|u(t)\| < \epsilon, \tag{1-20}$$

and there exists a γ with $0 < \gamma \leq \infty$, such that

$$2) \qquad \|u_0\| < \gamma \Rightarrow \lim_{t\to\infty} \|u(t)\| = 0. \tag{1-21}$$

If $\gamma = \infty$, we say the null solution is unconditionally nonlinearly stable (or simply refer to it as being nonlinearly asymptotically stable), otherwise for $\gamma < \infty$ the solution is conditionally stable. The value of γ is called the size of the attracting radius.

When the operator L is symmetric (or say self-adjoint), linear stability always implies nonlinear stability (the converse is also true).

The linear theory of hydrodynamic stability, however, deals with stability against infinitesimal disturbances. Generally, the basic flow is assumed to be steady and equations of motion and boundary conditions are linearized for sufficiently small disturbances. This results in a linear homogeneous system of partial differential equations and boundary conditions. Experience with the method of separation of variables and with Laplace transformations suggests that, in general, the solution of the system can be expressed as $u(x,t) = u(x)e^{-st}$ with $s = \sigma + i\omega$. The linear system then determines the values of s, called eigenvalues, and corresponding function $u(x)$, called eigenfunctions. This method, in which the disturbances are resolved into modes, is usually called the method of normal modes. If $\sigma > 0$ the disturbance to the basic flow is said to be asymptotically stable or stable, if $\sigma = 0$ the mode is said to be neutrally or marginally stable, and if $\sigma < 0$ it is said to be unstable. At marginal stability, if $\sigma = 0$ implies $\omega = 0$, then one says that the principle of the exchange of stability is valid.

Chapter 2

A Cartesian-Tensor Solution of Brinkman's Equation

1. Introduction.

Exact solutions of fluid flow problems in most of continuum fluid mechanics theories are rare. In the case of viscous fluid, even if we neglect inertial effects, obtaining an exact solution of the Stokes' equation for arbitrary body shapes or complicated flow conditions is recognized to be a difficult problem. Thus a variety of methods, such as the boundary-value method, singularity method and stream function method (see Niefer and Kaloni, 1986) have been proposed in the literature. When exact solutions are not possible, approximate solutions via numerical methods, perturbation methods, or some analytical techniques are commonly encountered. However, any attempt at searching for an exact general solution of any useful equation, of course, is natural and a valuable step whenever it is possible.

In this chapter we develop a general solution of the Brinkman equation, based on the use of Cartesian tensors. In essence it is an extension of the method developed earlier by Neifer and Kaloni (1986) to study the viscous creeping-motion equation. We have further developed this method to treat the viscous flow problem with more complicated non-spherical geometries (Qin and Kaloni 1990). It is believed that the solution obtained by the present method will be directly applicable to porous-flow problems for which the boundary conditions are given in the Cartesian-tensor form and to those that can be easily written in this manner. An integral-equation approach, making use of Green's function for the Brinkman equation, has been discussed by Higdon and Kojima (1981). In the next section we develop the solution based on the use of arbitrary, spatially constant, second- and third-order tensors. In this manner we are able to determine the general expressions for the velocity and pressure field directly from the developed solution. We remark that even though we have found the present solution to be sufficiently general to solve a number of problems, this solution can be generalized by using higher order terms.

To illustrate the use of the method, we give two applications in sections 3 and 4 of this chapter. We first apply it to find the drag on a porous sphere in a viscous fluid flow in section 3. The creeping flow past a porous spherical shell is then studied by using Stokes' and Brinkman' equations in section 4. Both examples show that the solution generated by this method is simple and useful. Moreover, having the exact solutions obtained by using Stokes' and Brinkman's equations with well-defined stress and velocity boundary conditions provides a better understanding in the porous-flow problems, for instance, for boundary layer effect. In the literature there is a widespread belief that, in many cases, the effect of the Brinkman term is negligible. Contrary to this belief the solutions and applications given below show that, even though Darcy's solution is easily obtained from the Brinkman solution by the proper approximation, there are several novel features in the solution which cannot be seen from Darcy's solution. We illustrate some of these remarks in section 5. Finally, we point out that in several limiting cases, the results obtained by previous authors are easily recovered from our work.

2. Basic Equations and Their Solution.

We consider the velocity field satisfying the continuity equation

$$\nabla \cdot \mathbf{u} = 0, \tag{2-2-1}$$

and the Brinkman equation

$$-\nabla \hat{p} + \mu \nabla^2 \mathbf{u} = \frac{\mu}{k}\mathbf{u}. \tag{2-2-2}$$

Here $\mathbf{u}$ is the volume average velocity, $\hat{p}$ the interstitial average pressure, μ the viscosity of the fluid and k the permeability of the porous medium. We note that, following Brinkman (1947), we have set $\mu^* = \mu$, which is considered to be a reasonable approximation. The above are four partial differential equations for the four unknowns $\mathbf{u}$ and $\hat{p}$. Taking the divergence of equation (2-2-2) and using (2-2-1), it turns out that solving the above system is equivalent to finding the solution of the equations

$$\nabla^2 p = 0\,, \tag{2-2-3}$$

$$\nabla^2 \mathbf{u} - C^2 \mathbf{u} = \nabla p\,, \tag{2-2-4}$$

where $p = \mu^{-1}\hat{p}$ and $C^2 = k^{-1}$. Once $\mathbf{u}$ is so determined we require it to satisfy equation (2-2-1).

We now generate solutions of the above system of equations which involve spatially-constant second- and third-order tensors a_{ij} and b_{ijk}, respectively. Following Niefer and Kaloni (1986), we write the scalar invariants linear in a_{ij} and b_{ijk}, in combination with the position vector x_i ($r^2 = x_i x_i$), as

$$a_{ij}, \quad \epsilon_{ijk} a_{kj} x_i, \quad a_{ij} x_i x_j, \quad b_{imm} x_i, \quad b_{mim} x_i \quad b_{mmi} x_i, \quad b_{ijk} x_i x_j x_k\,. \tag{2-2-5}$$

We take

$$\begin{aligned} p(r) = {} & H^0(r) a_{ii} + H^1(r) \epsilon_{ijk} a_{jk} x_i + H^2(r) b_{imm} x_i + H^3(r) b_{mim} x_i \\ & + H^4(r) b_{mmi} x_i + H^5(r) a_{ij} x_i x_j + H^6(r) b_{ijk} x_i x_j x_k\,. \end{aligned} \tag{2-2-6}$$

Substituting into (2-2-3), it is found that (also see Niefer and Kaloni, 1986)

$$
\begin{aligned}
H^0(r) &= -\frac{1}{3}B_1^5 r^{-3} + B_1^0 r^{-1} + A_1^0 - \frac{1}{3}A_1^5 r^2 , \\
H^1(r) &= B_1^1 r^{-3} + A_1^1 , \\
H^2(r) &= -\frac{1}{5}B_1^6 r^{-5} + B_1^2 r^{-3} + A_2^2 - \frac{1}{5}A_1^6 r^2 , \\
H^3(r) &= -\frac{1}{5}B_1^6 r^{-5} + B_1^3 r^{-3} + A_2^3 - \frac{1}{5}A_1^6 r^2 , \\
H^4(r) &= -\frac{1}{5}B_1^6 r^{-5} + B_1^4 r^{-3} + A_2^4 - \frac{1}{5}A_1^6 r^2 , \\
H^5(r) &= B_1^5 r^{-5} + A_1^5 , \\
H^6(r) &= B_1^6 r^{-7} + A_1^6 ,
\end{aligned}
\tag{2-2-7}
$$

where A_j^i and B_j^i are arbitrary constants.

Equations (2-2-4) and (2-2-6) suggest that the component of $\mathbf{u}$ in the direction of x_p can be assumed to be of the form

$$
\begin{aligned}
u_p(r) = {} & h_1^0(r) a_{ii} x_p + h_1^1(r)\epsilon_{ijk} a_{kj} x_i x_p + h_2^1(r)\epsilon_{pjk} a_{kj} + h_1^2(r) b_{imm} x_i x_p \\
& + h_2^2(r) b_{pmm} + h_1^3(r) b_{mim} x_i x_p + h_2^3(r) b_{mpm} + h_1^4(r) b_{mmi} x_i x_p \\
& + h_2^4(r) b_{mmp} + h_1^5(r) a_{ij} x_i x_j x_p + h_2^5(r) a_{pj} x_j + h_3^5(r) a_{jp} x_j \\
& + h_1^6(r) b_{ijk} x_i x_j x_k x_p + h_2^6(r) b_{pjk} x_j x_k + h_3^6(r) b_{jpk} x_j x_k + h_4^6(r) b_{jkp} x_j x_k .
\end{aligned}
\tag{2-2-8}
$$

Substitution of (2-2-8) and (2-2-6) into equation (2-2-4) yields the following type of differential equations:

$$
\begin{aligned}
(h_1^0)'' + \frac{4}{r}(h_1^0)' - C^2 h_1^0 &= \frac{1}{r}(H^0)' - 2h_1^5\,, \\
(h_1^1)'' + \frac{6}{r}(h_1^1)' - C^2 h_1^1 &= \frac{1}{r}(H^1)'\,, \\
(h_2^1)'' + \frac{2}{r}(h_2^1)' - C^2 h_2^1 &= H^1 - 2h_1^1\,, \\
(h_2^2)'' + \frac{2}{r}(h_2^2)' - C^2 h_2^2 &= H^2 - 2h_1^2 - 2h_1^6\,, \\
&\cdots \\
(h_4^6)'' + \frac{6}{r}(h_4^6)' - C^2 h_4^6 = H^6 - 2h_1^6, & \qquad (2\text{-}2\text{-}9)
\end{aligned}
$$

where the dashes denote differentiation with respect to the variable r.

We now consider the following equation,

$$y''(r) + \frac{2l}{r}y'(r) - C^2 y(r) = 0 \qquad (l = 1,2,3,4,5), \qquad (2\text{-}2\text{-}10)$$

and state some properties of solutions of (2-2-10) which are useful to construct the solutions of equation (2-2-9).

Property 1. *If $y_l(r)$ satisfies (2-2-10), then $y_{l+1}(r) = (1/r)(y_l)'$ satisfies*

$$y'' + \frac{2(l+1)}{r}y' - C^2 y = 0. \qquad (2\text{-}2\text{-}11)$$

Property 2. *If $y_{l+1}(r)$ satisfies equation (2-2-11), then $y_{l+1}(r)$ also satisfies the inhomogeneous equations*

$$y'' + \frac{2l}{r}y' - C^2 y = -2y_{l+2} \qquad (2\text{-}2\text{-}12)$$

and

$$y'' + \frac{2(l-1)}{r}y' - C^2 y = -4y_{l+2}\,, \qquad l \geq 2, \qquad (2\text{-}2\text{-}13)$$

where $y_{l+2} = \frac{1}{r} y'_{l+1}$.

The proof of the above properties can be verified by substitution of the appropriate expressions into the above equations and rearranging the resulting terms.

Let $\rho = Cr$ and $y = u\rho^{-l+1/2}$. Equation (2-2-10) then becomes a modified Bessel equation,

$$\rho^2 \frac{d^2u}{d\rho^2} + \rho \frac{du}{d\rho} - [\rho^2 + (l - \frac{1}{2})^2]u = 0, \tag{2-2-14}$$

and admits $I_{l-1/2}(\rho)$ and $I_{-l+1/2}(\rho)$ as two linearly independent solutions. Therefore, it follows that

$$\begin{aligned} y_l &= C^{2l-1}(Cr)^{-l+1/2} I_{l-1/2}(Cr), \\ y_l^* &= C^{2l-1}(Cr)^{-l+1/2} I_{-l+1/2}(Cr), \end{aligned} \tag{2-2-15}$$

are two linearly independent solutions of (2-2-10).

Property 3. *For y_l and y_l^* as defined in (2-2-15), we have*

$$y_{l+1} = \frac{1}{r}\frac{d}{dr} y_l(r), \qquad y_{l+1}^* = \frac{1}{r}\frac{d}{dr} y_l^*(r), \tag{2-2-16}$$

and

$$\begin{aligned} r\frac{d}{dr} y_{l+1}(r) + (2l+1) y_{l+1}(r) &= C^2 y_l(r), \\ r\frac{d}{dr} y_{l+1}^*(r) + (2l+1) y_{l+1}^*(r) &= C^2 y_l^*(r). \end{aligned} \tag{2-2-17}$$

The verification of the above properties, by use of the properties of Bessel functions, is straightforward.

Making use of the above properties, we now write down the solutions of equations (2-2-9):

$$h_1^0(r) = -\frac{1}{C^2} B_1^5 r^{-5} + \frac{1}{C^2} B_1^0 r^{-3} + \frac{2}{3C^2} A_1^5 + A_2^5 y_3(r) + A_2^0 y_2(r)$$

$$
\begin{aligned}
& + B_2^5 y_3^*(r) + B_2^0 y_2^*(r), \\
h_1^1(r) &= \frac{3}{C^2} B_1^1 r^{-5} + A_2^1 y_3(r) + B_2^1 y_3^*(r), \\
h_2^1(r) &= -\frac{1}{C^2} B_1^1 r^{-3} - \frac{1}{C^2} A_1^1 + A_2^1 y_2(r) + A_3^1 y_1(r) + B_2^1 y_2^*(r) + B_3^1 y_1^*(r), \\
h_1^2(r) &= -\frac{1}{C^2} B_1^6 r^{-7} + \frac{3}{C^2} B_1^2 r^{-5} + \frac{2}{5C^2} A_1^6 + A_2^6 y_4(r) + A_3^2 y_3(r) \\
& \quad + B_2^6 y_4^*(r) + B_3^2 y_3^*(r), \\
h_2^2(r) &= \frac{1}{5C^2} B_1^6 r^{-5} - \frac{1}{C^2} B_1^2 r^{-3} - \frac{1}{C^2} A_2^2 + \frac{1}{5C^2} A_1^6 r^2 + A_2^6 y_3(r) \\
& \quad + (A_3^2 + A_3^6) y_2(r) + A_4^2 y_1(r) + B_2^6 y_3^*(r) + (B_3^2 + B_3^6) y_2^*(r) + B_4^2 y_1^*(r), \\
h_1^3(r) &= -\frac{1}{C^2} B_1^6 r^{-7} + \frac{1}{C^2} B_1^3 r^{-5} + \frac{2}{5C^2} A_1^6 + A_2^6 y_4(r) + A_3^3 y_3(r) \\
& \quad + B_2^6 y_4^*(r) + B_3^3 y_3^*(r), \\
h_2^3(r) &= \frac{1}{5C^2} B_1^6 r^{-5} - \frac{1}{C^2} B_1^3 r^{-3} - \frac{1}{C^2} A_2^3 + \frac{1}{5C^2} A_1^6 r^2 + A_2^6 y_3(r) \\
& \quad + (A_3^3 + A_4^6) y_2(r) + A_4^3 y_1(r) + B_2^6 y_3^*(r) + (B_3^3 + B_4^6) y_2^*(r) + B_4^3 y_1^*(r), \\
h_1^4(r) &= \frac{1}{C^2} B_1^6 r^{-7} + \frac{3}{C^2} B_1^4 r^{-5} + \frac{2}{5C^2} A_1^6 + A_2^6 y_4(r) + A_3^4 y_3(r) \\
& \quad + B_2^6 y_4^*(r) + B_3^4 y_3^*(r), \\
h_2^4(r) &= \frac{1}{5C^2} B_1^6 r^{-5} - \frac{1}{C^2} B_1^4 r^{-3} - \frac{1}{C^2} A_2^4 + \frac{1}{5C^2} A_1^6 r^2 + A_2^6 y_3(r) \\
& \quad + (A_3^4 + A_5^6) y_2(r) + A_4^4 y_1(r) + B_2^6 y_3^*(r) + (B_3^4 + B_5^6) y_2^*(r) + B_4^4 y_1^*(r), \\
h_1^5(r) &= \frac{5}{C^2} B_1^5 r^{-7} + A_2^5 y_4(r) + B_2^5 y_4^*(r), \\
h_2^5(r) &= -\frac{1}{C^2} B_1^5 r^{-5} - \frac{1}{C^2} A_1^5 + A_2^5 y_3(r) + A_3^5 y_2(r) + B_2^5 y_3^*(r) + B_3^5 y_2^*(r), \\
h_3^5(r) &= -\frac{1}{C^2} B_1^5 r^{-5} - \frac{1}{C^2} A_1^5 + A_2^5 y_3(r) + A_4^5 y_2(r) + B_2^5 y_3^*(r) + B_4^5 y_2^*(r), \\
h_1^6(r) &= \frac{7}{C^2} B_1^6 r^{-9} + A_2^6 y_5(r) + B_2^6 y_5^*(r), \\
h_2^6(r) &= -\frac{1}{C^2} B_1^6 r^{-7} - \frac{1}{C^2} A_1^6 + A_2^6 y_4(r) + A_3^6 y_3(r) + B_2^6 y_4^*(r) + B_3^6 y_3^*(r), \\
h_3^6(r) &= -\frac{1}{C^2} B_1^6 r^{-7} - \frac{1}{C^2} A_1^6 + A_2^6 y_4(r) + A_4^6 y_3(r) + B_2^6 y_4^*(r) + B_4^6 y_3^*(r),
\end{aligned}
$$

$$h_4^6(r) = -\frac{1}{C^2}B_1^6 r^{-7} - \frac{1}{C^2}A_1^6 + A_2^6 y_4(r) + A_5^6 y_3(r) + B_2^6 y_4^*(r) + B_5^6 y_3^*(r). \tag{2-2-18}$$

The equation of continuity imposes the following restrictions upon the arbitrary constants:

$$\begin{aligned}
A_2^0 &= 0, & B_2^0 &= 0, \\
A_2^5 &= \frac{A_3^5 + A_4^5}{C^2}, & B_2^5 &= \frac{B_3^5 + B_4^5}{C^2}, \\
A_2^1 &= -\frac{A_3^1}{C^2}, & B_2^1 &= -\frac{B_3^1}{C^2}, \\
A_2^6 &= -\frac{A_3^6 + A_4^6 + A_5^6}{C^2}, & B_2^6 &= -\frac{B_3^6 + B_4^6 + B_5^6}{C^2}, \\
A_3^2 &= -\frac{A_4^2}{C^2}, & B_3^2 &= -\frac{B_4^2}{C^2}, \\
A_3^3 &= -\frac{A_4^3}{C^2}, & B_3^3 &= -\frac{B_4^3}{C^2}, \\
A_3^4 &= -\frac{A_4^4}{C^2}, & B_3^4 &= -\frac{B_4^4}{C^2}.
\end{aligned} \tag{2-2-19}$$

When the combinations in (2-2-19) are introduced into (2-2-18), we find the final form of the solutions to be

$$\begin{aligned}
h_1^0(r) &= -\frac{1}{C^2}B_1^5 r^{-5} + \frac{1}{C^2}B_1^0 r^{-3} + \frac{2}{3C^2}A_1^5 - \frac{1}{C^2}(A_3^5 + A_4^5)y_3(r) \\
&\quad - \frac{1}{C^2}(B_3^5 + B_4^5)y_3^*(r), \\
h_1^1(r) &= \frac{3}{C^2}B_1^1 r^{-5} - \frac{1}{C^2}A_3^1 y_3(r) - \frac{1}{C^2}B_3^1 y_3^*(r), \\
h_2^1(r) &= -\frac{1}{C^2}B_1^1 r^{-3} - \frac{1}{C^2}A_1^1 - \frac{1}{C^2}A_3^1 y_2(r) + A_3^1 y_1(r) - \frac{1}{C^2}B_3^1 y_2^*(r) + B_3^1 y_1^*(r), \\
h_1^2(r) &= -\frac{1}{C^2}B_1^6 r^{-7} + \frac{3}{C^2}B_1^2 r^{-5} + \frac{2}{5C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) \\
&\quad - \frac{1}{C^2}A_4^2 y_3(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) - \frac{1}{C^2}B_4^2 y_3^*(r), \\
h_2^2(r) &= \frac{1}{5C^2}B_1^6 r^{-5} - \frac{1}{C^2}B_1^2 r^{-3} - \frac{1}{C^2}A_2^2 + \frac{1}{5C^2}A_1^6 r^2 \\
&\quad - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_3(r) + (-\frac{1}{C^2}A_4^2 + A_3^6)y_2(r) + A_4^2 y_1(r)
\end{aligned}$$

$$
\begin{aligned}
& - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_3^*(r) + (-\frac{1}{C^2}B_4^2 + B_3^6)y_2^*(r) + B_4^2 y_1^*(r), \\
h_1^3(r) &= -\frac{1}{C^2}B_1^6 r^{-7} + \frac{3}{C^2}B_1^3 r^{-5} + \frac{2}{5C^2}A_1^6 \\
& - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) - \frac{1}{C^2}A_4^3 y_3(r) \\
& - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) - \frac{1}{C^2}B_4^3 y_3^*(r), \\
h_2^3(r) &= \frac{1}{5C^2}B_1^6 r^{-5} - \frac{1}{C^2}B_1^3 r^{-3} - \frac{1}{C^2}A_2^3 + \frac{1}{5C^2}A_1^6 r^2 \\
& - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_3(r) + (A_4^6 - \frac{1}{C^2}A_4^3)y_2(r) + A_4^3 y_1(r) \\
& - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_3^*(r) + (B_4^6 - \frac{1}{C^2}B_4^3)y_2^*(r) + B_4^3 y_1^*(r), \\
h_1^4(r) &= -\frac{1}{C^2}B_1^6 r^{-7} + \frac{3}{C^2}B_1^4 r^{-5} + \frac{2}{5C^2}A_1^6 \\
& - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) - \frac{1}{C^2}A_4^4 y_3(r) \\
& - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) - \frac{1}{C^2}B_4^4 y_3^*(r). \\
h_2^4(r) &= \frac{1}{5C^2}B_1^6 r^{-5} - \frac{1}{C^2}B_1^4 r^{-3} - \frac{1}{C^2}A_2^4 + \frac{1}{5C^2}A_1^6 r^2 \\
& - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_3(r) + (A_5^6 - \frac{1}{C^2}A_4^4)y_2(r) + A_4^4 y_1(r) \\
& - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_3^*(r) + (B_5^6 - \frac{1}{C^2}B_4^4)y_2^*(r) + B_4^4 y_1^*(r), \\
h_1^5(r) &= \frac{5}{C^2}B_1^5 r^{-7} - \frac{1}{C^2}(A_3^5 + A_4^5)y_4(r) - \frac{1}{C^2}(B_3^5 + B_4^5)y_4^*(r), \\
h_2^5(r) &= -\frac{1}{C^2}B_1^5 r^{-5} - \frac{1}{C^2}A_1^5 - \frac{1}{C^2}(A_3^5 + A_4^5)y_3(r) + A_3^5 y_2(r) \\
& - \frac{1}{C^2}(B_3^5 + B_4^5)y_3^*(r) + B_3^5 y_2^*(r), \\
h_3^5(r) &= -\frac{1}{C^2}B_1^5 r^{-5} - \frac{1}{C^2}A_1^5 - \frac{1}{C^2}(A_3^5 + A_4^5)y_3(r) + A_4^5 y_2(r) \\
& - \frac{1}{C^2}(B_3^5 + B_4^5)y_3^*(r) + B_4^5 y_2^*(r), \\
h_1^6(r) &= \frac{7}{C^2}B_1^6 r^{-9} - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_5(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_5^*(r), \\
h_2^6(r) &= -\frac{1}{C^2}B_1^6 r^{-7} - \frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r) \\
& + A_3^6 y_3(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) + B_3^6 y_3^*(r),
\end{aligned}
$$

$$h_3^6(r) = -\frac{1}{C^2}B_1^6 r^{-7} - \frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r)$$
$$+ A_4^6 y_3(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) + B_4^6 y_3^*(r),$$
$$h_4^6(r) = -\frac{1}{C^2}B_1^6 r^{-7} - \frac{1}{C^2}A_1^6 - \frac{1}{C^2}(A_3^6 + A_4^6 + A_5^6)y_4(r)$$
$$+ A_5^6 y_3(r) - \frac{1}{C^2}(B_3^6 + B_4^6 + B_5^6)y_4^*(r) + B_5^6 y_3^*(r). \tag{2-2-20}$$

In the above equations, the functions y_i and y_i^* $(i = 1, 2, 3, \dots)$ are defined through equation (2-2-15). We remark that constraints (2-2-19) do not suggest any changes in the form of equation (2-2-7) for the pressure.

3. Hydrodynamic Force on a Porous Sphere.

As the first application of the method described in the previous section, we now consider the flow of a viscous fluid past a porous sphere. A solution to this problem when the flow, both inside and outside of the sphere, is approximated by Stokes' equation has been considered by Lenov (1962). It is, however, now believed that in order to describe more accurately the flow inside the porous sphere, Stokes' equation should be replaced by Brinkman's equation. Theoretical justification supporting this view has been proposed by Howells (1974), Hinch (1977), and many others. On the other hand, the indiscriminate use of this equation for all situations has also been questioned. Here we consider the fluid outside the sphere to be a Stokes' flow with undisturbed velocity U far from the sphere. Inside the sphere we assume that the average velocity and pressure satisfy the Brinkman equation (2-2-2). With the boundary conditions that both the velocity and surface forces be continuous across the surface of the porous sphere we determine the velocity and pressure distributions both inside and outside the sphere. We then use these quantities to calculate the drag on the sphere.

For the Stokes flow outside the sphere we use the solution obtained in Niefer and

Kaloni (1986) and select the Cartesian-tensor form of the boundary conditions as

$$p_\infty = a_{ii}, \qquad u_{l\infty} = \epsilon_{ljk}a_{kj} = U\mathbf{e}_3. \tag{2-3-1}$$

It then follows that (cf. Eqs. (20)-(22) of Niefer and Kaloni, 1986)

$$p^{(o)} = p_\infty + A_1^1 r^{-1} U x_3, \tag{2-3-2}$$

$$u_l^{(o)} = (A_3^1 r^{-5} + \frac{1}{2}A_1^1 r^{-3})U x_3 x_l + (1 - \frac{1}{3}A_3^1 r^{-3} + \frac{1}{2}A_1^1 r^{-1})U\delta_{l3}, \quad (l = 1,2,3). \tag{2-3-3}$$

where we have added the superscript (o) to denote quantities *outside* of the porous sphere. For the fluid *inside* the sphere we will use the superscript (i). For the flow inside the porous sphere we select

$$p^{(i)} = \bar{A}_1^1 U x_3, \tag{2-3-4}$$

$$u_l^{(i)} = -\frac{1}{C^2}\bar{A}_3^1 y_3(r) U x_3 x_l + [-\frac{1}{C^2}\bar{A}_1^1 - \frac{1}{C^2}\bar{A}_3^1 y_2(r) + \bar{A}_3^1 y_1(r)]U\delta_{l3}, \quad (l = 1,2,3). \tag{2-3-5}$$

Redefining the four constants appearing in the above equations as

$$A = A_3^1, \qquad B = A_1^1, \qquad D = \bar{A}_1^1, \qquad E = \bar{A}_3^1, \tag{2-3-6}$$

we can write the expressions in component form:

$$\begin{aligned}
p^{(o)} &= Br^{-3}Ux_3,\\
u_r^{(o)} &= (\frac{2}{3}Ar^{-3} + Br^{-1} + 1)U\cos\theta,\\
u_\theta^{(o)} &= (\frac{1}{3}Ar^{-3} - \frac{1}{2}Br^{-1} - 1)U\sin\theta,\\
u_\phi^{(o)} &= 0,\\
p^{(i)} &= DUx_3,\\
u_r^{(i)} &= [-\frac{1}{C^2}D + \frac{2}{C^2}Ey_2(r)]U\cos\theta,\\
u_\theta^{(i)} &= [\frac{1}{C^2}D + \frac{1}{C^2}Ey_2(r) - Ey_1(r)]U\sin\theta,\\
u_\phi^{(i)} &= 0.
\end{aligned} \tag{2-3-7}$$

Applying the boundary conditions that the velocity and stress components be continuous across the surface of the porous sphere ($r = a$) we get

$$
\begin{aligned}
\frac{2}{3}A + Ba^2 + a^3 &= \frac{1}{C^2}Da^3 + \frac{2}{C^2}Ea^3y_2(a), \\
\frac{1}{3}A - \frac{1}{2}Ba^2 - a^3 &= \frac{1}{C^2}Da^3 + \frac{1}{C^2}Ea^3y_2(a) - Ea^3y_1(a), \\
-4A - 3Ba^2 &= -Da^5 + \frac{4}{C^2}Ea^5y_3(a), \\
\frac{5}{3}A - \frac{1}{2}Ba^2 - a^3 &= \frac{1}{C^2}Da^3 + \frac{1}{C^2}Ea^5y_3(a) \\
&\quad - \frac{2}{C^2}Ea^3y_2(a) - Ea^5y_1(a).
\end{aligned}
\tag{2-3-8}
$$

Solving this system of equations we obtain

$$
\begin{aligned}
A &= [\frac{3}{C^2}a^3y_2(a) + \frac{1}{2}a^5y_2(a) - a^3y_1(a)]E, \\
B &= -a^3y_2(a)E, \\
D &= -y_2(a)E, \\
E &= 3a^3[\frac{3}{C^2}a^3y_2(a) + 2a^5y_2(a) + 2a^3y_1(a)]^{-1}.
\end{aligned}
\tag{2-3-9}
$$

We note that in writing (2-3-9) we have used the result that $a^5y_3(a) + 3a^3y_2(a) = C^2a^2y_1(a)$. Furthermore, we also want to emphasize that a variation in boundary conditions will result in different values of the constants. When the constants determined in (2-3-9) are employed in (2-3-7), we get the complete expressions for the pressure and the velocity distribution.

In order to determine the drag on the porous sphere, which will be directed along the axis of symmetry, we need to calculate

$$
D = 2\pi a^2 \int_0^{\pi} (t_{rr}\cos\theta - t_{r\theta}\sin\theta)\sin\theta\, d\theta. \tag{2-3-10}
$$

On calculating the expressions for the stresses in spherical polar coordinates by using (2-3-6), we find

$$
D = -4\pi\mu UB, \tag{2-3-11}
$$

where

$$B^{-1} = -\frac{E^{-1}}{a^3 y_2(a)} = -\frac{1}{3a^6 y_2(a)}[\frac{3}{C^2}a^3 y_2(a) + 2a^5 y_2(a) + 2a^3 y_1(a)]. \quad \text{(2-3-12)}$$

Equations (2-3-11) and (2-3-12) thus give

$$D = 12a^6 \pi\mu U[\frac{3}{C^2}a^3 + 2a^5 + 2a^3\frac{y_1(a)}{y_2(a)}]^{-1}. \quad \text{(2-3-13)}$$

It is of some interest to determine asymptotic values of the drag D in equation (2-3-13) for low and high values of the permeability parameter $k (= 1/C^2)$. For large permeability we can write $y_1(a)/y_2(a)$ in (2-3-13) as

$$\frac{y_1(a)}{y_2(a)} = \left[\frac{1}{a}\sinh(ca)\right] / \left[C\frac{\cosh(Ca)}{a^2} - \frac{\sinh(Ca)}{a^3}\right]$$
$$= \left[\frac{C}{a}\coth(Ca) - \frac{1}{a^2}\right]^{-1} \approx \frac{3}{C^2} + \frac{3a^2}{15} = 3(k + \frac{a^2}{15}). \quad \text{(2-3-14)}$$

Substitution of (2-3-14) into (2-3-13) and further simplification produces

$$D = 4\mu\frac{\pi U a}{3}\left[\frac{a^2}{k} - \frac{4}{15}\frac{a^4}{k^2} + O\left(\frac{a^6}{k^3}\right)\right]. \quad \text{(2-3-15)}$$

For low permeability we can approximate $y_1(a)/y_2(a)$ by

$$\frac{y_1(a)}{y_2(a)} \approx \frac{a}{C} = a\sqrt{k}, \quad \text{(2-3-16)}$$

and in this case (2-3-13) gives

$$D = 6\mu\pi U a\left[1 - \frac{\sqrt{k}}{a} + O\left(\frac{k}{a^2}\right)\right]. \quad \text{(2-3-17)}$$

Clearly, when $k \to 0$, i.e. $C \to \infty$, the above reduces to the classical value determined by Stokes (Landau and Lifshitz, 1956), since in that case the porous sphere behaves like a solid sphere and $u_l^{(i)} = 0$. For nonzero values of k, it is found that the drag on a porous sphere is lower than that on a solid sphere and that it decreases when k increases.

We point out that both of the above expressions agree with the results of Higdon and Kojima (1981) who derived them by solving integral equations.

4. Creeping Flow Past a Porous Spherical Shell.

Now we consider the problem of a porous spherical shell, of external radius a and internal radius b, immersed in a uniform stream of a viscous fluid with velocity U. Following Jones (1973), we shall denote the external region, porous region and cavity as regions I, II, and III, respectively. Thus the equations to be solved are

$$\begin{aligned} \mu_1 \nabla^2 \mathbf{u}^{(1)} &= \nabla p^{(1)} \qquad \text{in } I, \\ \hat{\mu}_2 \nabla^2 \mathbf{v} - \frac{\mu_2}{k}\mathbf{v} &= \nabla \hat{p} \qquad \text{in } II, \\ \mu_3 \nabla^2 \mathbf{u}^{(3)} &= \nabla p^{(3)} \qquad \text{in } III, \end{aligned} \tag{2-4-1}$$

along with the conservation of mass

$$\nabla \cdot \mathbf{u}^{(1)} = 0, \qquad \nabla \cdot \mathbf{v} = 0, \qquad \nabla \cdot \mathbf{u}^{(3)} = 0,$$

and boundary conditions

$$u_r^{(1)}(a,\theta) = v_r(a,\theta), \qquad u_\theta^{(1)}(a,\theta) = v_\theta(a,\theta), \qquad u_r^{(3)}(b,\theta) = v_r(b,\theta),$$

$$u_\theta^{(3)}(b,\theta) = v_\theta(b,\theta), \qquad \tau_{r\theta}^{(1)}(a,\theta) = T_{r\theta}(a,\theta), \qquad \tau_{r\theta}^{(3)}(b,\theta) = T_{r\theta}(b,\theta),$$

$$p^{(1)}(a,\theta) = \hat{p}(a,\theta), \qquad p^{(3)}(b,\theta) = \hat{p}(b,\theta), \tag{2-4-2}$$

$$\begin{aligned} lim_{r\to\infty} u_r^{(1)}(r,\theta) &= U\cos\theta, \\ lim_{r\to\infty} u_\theta^{(1)}(r,\theta) &= -U\sin\theta, \end{aligned}$$

where τ and $\mathbf{T}$ are stresses in the viscous fluid and porous media, respectively. We note that the last condition in (2-4-2) suggests writing

$$p_\infty^{(1)} = 0, \qquad u_{\ell,\infty}^{(1)} = \epsilon_{\ell jk} a_{kj} = U\mathbf{e}_3. \tag{2-4-3}$$

In view of (2-4-3) we can write the expressions for the velocity components and pressure in the three different regions as

$$
\begin{aligned}
u_\ell^{(1)} &= (B_3^1 r^{-5} + \frac{1}{2}B_1^1 r^{-3})Ux_3x_\ell + (1 - \frac{1}{3}B_3^1 r^{-3} + \frac{1}{2}B_1^1 r^{-1})U\delta_{\ell 3}, \\
p^{(1)} &= \mu_1 B_1^1 r^{-3} U x_3, \qquad (2\text{-}4\text{-}4) \\
u_\ell^{(3)} &= -\frac{1}{10}A_2^1 U x_3 x_\ell + (A_6^1 + \frac{1}{5}A_2^1 r^2)U\delta_{\ell 3}, \\
p^{(3)} &= \mu_3 A_2^1 U x_3, \qquad (2\text{-}4\text{-}5) \\
v_\ell &= [\frac{3}{C^2}D_1^1 r^{-5} - \frac{1}{C^2}C_3^1 y_3(r) - \frac{1}{C^2}D_3^1 y_3^*(r)]Ux_3x_\ell \\
&\quad + [-\frac{1}{C^2}D_1^1 r^{-3} - \frac{1}{C^2}C_1^1 - \frac{1}{C^2}C_3^1 y_2(r) + C_3^1 y_1(r) \\
&\quad - \frac{1}{C^2}D_3^1 y_2^*(r) + D_3^1 y_1^*(r)]U\delta_{\ell 3}, \\
\hat{p} &= \mu_2(D_1^1 r^{-3} + C_1^1)Ux_3, \qquad (2\text{-}4\text{-}6)
\end{aligned}
$$

where we have assumed that $\hat{\mu_2} = \mu_2$. On redefining the eight constants as

$$
\begin{aligned}
&B_1^1 = \alpha_1, \quad B_3^1 = \alpha_2, \quad A_2^1 = \beta_1, \quad A_6^1 = \beta_2, \\
&C_1^1 = \gamma_1, \quad D_1^1 = \gamma_2, \quad C_3^1 = \gamma_3, \quad D_3^1 = \gamma_4, \qquad (2\text{-}4\text{-}7)
\end{aligned}
$$

and applying the boundary conditions that velocity, pressure and stress components be continuous across the surface of the porous spherical envelope ($r = a$ and $r = b$), we get

$$
\begin{aligned}
a^3 + \alpha_1 a^2 + \frac{2}{3}\alpha_2 &= \frac{1}{C^2}[-\gamma_1 a^3 + 2\gamma_2 + 2\gamma_3 a^3 y_2(a) + 2\gamma_4 a^3 y_2^*(a)], \\
-a^3 - \frac{1}{2}\alpha_1 a^2 + \frac{1}{3}\alpha_2 &= \frac{1}{C^2}[\gamma_1 a^3 + \gamma_2 + \gamma_3\{a^3 y_2(a) - C^2 a^3 y_1(a)\} \\
&\quad + \gamma_4\{a^3 y_2^*(a) - C^2 a^3 y_1^*(a)\}], \\
\frac{1}{10}\beta_1 b^5 + \beta_2 b^3 &= \frac{1}{C^2}[-\gamma_1 b^3 + 2\gamma_2 + 2\gamma_3 b^3 y_2(b) + 2\gamma_4 b^3 y_2^*(b)], \\
-(\frac{1}{5}\beta_1 b^5 + \beta_2 b^3) &= \frac{1}{C^2}[\gamma_1 b^3 + \gamma_2 + \gamma_3\{b^3 y_2(b) - C^2 b^3 y_1(b)\}
\end{aligned}
$$

$$+\gamma_4\{b^3y_2^*(b)-C^2b^3y_1^*(b)\}],$$

$$\frac{3}{5}\alpha_2+\frac{1}{2}\alpha_1a^2+a^3=-\frac{1}{C^2}[\gamma_1a^3-5\gamma_2+\gamma_3\{C^2a^3y_1(a)-5a^3y_2(a)$$

$$-C^2a^5y_2(a)\}+\gamma_4\{C^2a^3y_1^*(a)-5a^3y_2^*(a)-C^2a^5y_2^*(a)\}],$$

$$-4\alpha_2-3\alpha_1a^2=\gamma_1a^5-\gamma_2(a^2+\frac{12}{C^2})+\frac{4}{C^2}\gamma_3a^5y_3(a)+\frac{4}{C^2}\gamma_4a^5y_3^*(a),$$

$$\frac{1}{2}\beta_1b^5-\beta_2b^3=\frac{\sigma}{C^2}[\gamma_1b^3-5\gamma_2+\gamma_3\{C^2b^3y_1(b)-5b^3y_2(b)$$

$$-C^2b^5y_2(B)\}+\gamma_4\{C^2b^3y_1^*(b)-5b^3y_2^*(b)-C^2b^5y_2^*(b)\}],$$

$$\frac{3}{5}\beta_1b^5=\sigma[-\gamma_1b^5-\gamma_2(b^2+\frac{12}{C^2})+\frac{4}{C^2}\gamma_3b^5y_3(b)+\frac{4}{C^2}\gamma_4b^5y_3^*(b)],$$

(2-4-8)

where $\sigma=(\mu_2/\mu_3)$ and where we have assumed $\mu_1=\mu_2$. Solving (2-4-8) it follows that

$$\gamma_1(2a^3+\frac{3}{C^2}a)-2\gamma_2+2\gamma_3ay_1(a)+2\gamma_4ay_1^*(a)=3a,$$

$$\gamma_1a^3+\gamma_2+\gamma_3a^3y_2(a)+\gamma_4a^3y_2^*(a)=0,$$

$$\gamma_1\{\frac{2b^2}{C^2}(1-\sigma)-b^5\sigma\}+\gamma_2\{\frac{2}{C^2}(1-\sigma)-b^2\sigma\}+\gamma_3h_1+\gamma_4h_1^*=o,$$

$$-\gamma_1(\frac{2}{5}\sigma b^5)+\gamma_2\{\frac{36}{5C^2}(\sigma-1)-\sigma(\frac{2}{5}b^2+\frac{12}{C^2})\}+\gamma_3h_2+\gamma_4h_2^*=0,$$

(2-4-9)

where

$$h_1=\{\frac{2b^3}{C^2}y_2(b)-2b^3y_1(b)\}(1-\sigma)+2\sigma b^5y_2(b),$$

$$h_1^*=\{\frac{2b^3}{C^2}y_2^*(b)-2b^3y_1^*(b)\}(1-\sigma)+2\sigma b^5y_2^*(b),$$

$$h_2=\{\frac{12b^3}{5}y_1(b)-\frac{36b^3}{5C^2}y_2(b)\}(1-\sigma)+\frac{4\sigma b^5}{C^2}y_3(b),$$

$$h_2^*=\{\frac{12b^3}{5}y_1^*(b)-\frac{36b^3}{5C^2}y_2^*(b)\}(1-\sigma)+\frac{4\sigma b^5}{C^2}y_3^*(b).$$

(2-4-10)

Once having solved for γ_i $(i=1,2,3,4)$ from (2-4-9) and (2-4-10), we can easily obtain $\alpha_1,\alpha_2,\beta_1$ and β_2. In particular we note that

$$\alpha_1=\gamma_1a^3+\gamma_2=-\gamma_3a^3y_2(a)-\gamma_4a^3y_2^*(a). \tag{2-4-11}$$

Even though it is not difficult to solve the above algebraic system of simultaneous equations, the resulting solutions appear to be unwieldy. We shall, therefore, consider some particular cases only:

Case I $\sigma = 1$. This is the case corresponding to the problem discussed by Jones (1973) using Darcy's law and slip boundary conditions. In the present case we set $\sigma = 1$ in (2-4-9) and define

$$\gamma_1 = \frac{D_1}{D}, \qquad \gamma_2 = \frac{D_2}{D}, \qquad \gamma_3 = \frac{D_3}{D}, \qquad \gamma_4 = \frac{D_4}{D}, \tag{2-4-12}$$

where D is the coefficient determinant while D_i $(i = 1, 2, 3, 4)$ in the numerator are the same determinant D but with the i-th column replaced by the elements from the right-hand sides of equations in (2-4-9). After considerable algebra we find

$$\begin{aligned}
D &= \frac{8\ell_1}{C^5\pi}\{-12\ell_2^3 + [27\ell_2 - 27\ell_1 - 18\ell_1^3 - \frac{18}{5}\ell_2^2\ell_1^3 - \frac{12}{5}\ell_2^3\ell_1^3 \\
&\quad + \frac{33}{5}\ell_2^3 + \frac{12}{5}\ell_2^5]\cosh\Delta + [27 - 27\ell_1\ell_2 - 18\ell_2\ell_1^3 \\
&\quad + 6\ell_2^4 + \frac{18}{5}\ell_2^2 - \frac{3}{5}\ell_2^3\ell_1 - \frac{2}{5}\ell_2^3\ell_1^3 + \frac{2}{5}\ell_2^6]\sinh\Delta\}, \\
D_1 &= \frac{24\ell_1}{C^3\pi}\{2\ell_2^3 + [\frac{6}{5}\ell_2^2\ell_1 - \frac{1}{5}\ell_2^3 + 9\ell_1 - 9\ell_2]\cosh\Delta \\
&\quad + [\frac{1}{5}\ell_2^3\ell_1 - \frac{6}{5}\ell_2^2 + 9\ell_1\ell_2 - 9]\sin\Delta\}, \\
D_2 &= -\frac{24\ell_1}{C^3\pi}\{2\ell_2^3\ell_1^3 + [\frac{6}{5}\ell_2^5\ell_1 - \frac{1}{5}\ell_2^6 + 3\ell_2^3\ell_1 - 3\ell_2^4]\cosh\Delta \\
&\quad + [\frac{1}{5}\ell_2^6\ell_1 - \frac{6}{5}\ell_2^5 + 3\ell_2^4\ell_1 - 3\ell_2^3]\sinh\Delta\}, \\
D_3 &= -\frac{12\ell_1}{C^6}\sqrt{\frac{2}{\pi}}\{-3\ell_2^3\ell_1\sinh\ell_1 + 3\ell_2^3\cosh\ell_1 + [\frac{6}{5}\ell_2^2\ell_1^3 - \frac{6}{5}\ell_2^5 \\
&\quad + 9\ell_1^3 - 3\ell_2^3]\cos\ell_2 + [\frac{1}{5}\ell_2^6 - \frac{1}{5}\ell_2^3\ell_1^3 + 3\ell_2^4 - 9\ell_2\ell_1^3]\sinh\ell_2\}, \\
D_4 &= \frac{12\ell_1}{C^6}\sqrt{\frac{2}{\pi}}\{3\ell_2^3\sinh\ell_1 - 3\ell_2^3\ell_1\cosh\ell_1 + [\frac{1}{5}\ell_2^6 - \frac{1}{5}\ell_2^3\ell_1^3 \\
&\quad + 3\ell_2^4 - 9\ell_2\ell_1^3]\cosh\ell_2 + [\frac{6}{5}\ell_2^2\ell_1^3 - \frac{6}{5}\ell_2^5 + 9\ell_1^3 - 3\ell_2^3]\sinh\ell_2\},
\end{aligned} \tag{2-4-12}$$

where

$$\ell_1 = Ca = \frac{a}{\sqrt{k}}, \qquad \ell_2 = Cb = \frac{b}{\sqrt{k}}, \qquad \triangle = (\ell_1 - \ell_2). \tag{2-4-13}$$

With the γ_i so determined, the values of α_1, α_2, β_1 and β_2 are easily determined. In particular we find

$$\begin{aligned}
\alpha_1 &= \gamma_1 a^3 + \gamma_2 = -\gamma_3 a^3 y_2(a) - \gamma_4 a^3 y_2^*(a), \\
\beta_1 &= 2\gamma_3 y_2(b) + 2\gamma_4 y_2^*(b), \\
2\alpha_2 &= \frac{1}{C^2}[6\gamma_2 + \gamma_3\{6a^3 y_2(a) + C^2 a^5 y_2(a) - 2C^2 a^3 y_1(a)\} \\
&\qquad + \gamma_4\{6a^3 y_2^*(a) + C^2 a^5 y_2^*(a) - 2C^2 a^3 y_1^*(a)\}], \\
\beta_2 b^3 &= \frac{1}{C^2}[-\gamma_1 b^3 + 5\gamma_2 + \gamma_3\{5b^3 y_2(b) - C^2 b^3 y_1(b)\} \\
&\qquad + \gamma_4\{5b^3 y_2^*(a) - C^2 b^3 y_1^*(b)\}].
\end{aligned} \tag{2-4-14}$$

When the values of α's, β's and γ's so determined are substituted into (2-4-4)-(2-4-6), we get the exact expressions for velocities and pressures in the three regions. These are thus the solutions of the Stokes-Brinkman equations. In the next section we shall show how we can recover Darcy's solution from these results.

The force on the porous shell is given by the formula $F = 2\pi a^2 \int_0^\pi (t_{rr}\cos\theta - t_{r\theta}\sin\theta)\sin\theta\, d\theta$ where t_{rr} and $t_{r\theta}$ are to be calculated from (2-4-4) and (2-4-5). On carrying out the integration it follows that

$$F = 12\frac{\pi\mu U}{C}\left(\frac{M}{N}\right) \tag{2-4-15}$$

where

$$
\begin{aligned}
M =& \{\frac{6}{5}(\ell_2^2\ell_1^4 - \ell_2^5\ell_1) + \frac{1}{5}(\ell_2^6 - \ell_1^3\ell_2^3) + 9\ell_1^4 - 9\ell_1^3\ell_2 \\
&+ 3\ell_2^4 - 3\ell_2^3\ell_1]\cosh\triangle + [\frac{1}{5}(\ell_2^3\ell_1^4 - \ell_2^6\ell_1) + \frac{6}{5}(\ell_2^5 - \ell_2^2\ell_1^3) \\
&+ 9\ell_2\ell_1^4 - 9\ell_1^3 + 3\ell_2^3 - 3\ell_2^4\ell_1]\sinh\triangle\}, \\
N =& \{12\ell_2^3 + [27\ell_1 - 27\ell_2 + 18\ell_1^3 + \frac{18}{5}\ell_2^2\ell_1 + \frac{12}{5}\ell_2^2\ell_1^2 \\
&- \frac{33}{5}\ell_2^3 - \frac{12}{5}\ell_2^5]\cosh\triangle + [27\ell_1\ell_2 - 27 + 18\ell_2\ell_1^3 - 6\ell_2^4 \\
&- \frac{18}{5}\ell_2^2 + \frac{3}{5}\ell_2^3\ell_1 + \frac{2}{5}\ell_2^3\ell_1^3 - \frac{2}{5}\ell_2^6]\sinh\triangle\}.
\end{aligned}
\tag{2-4-16}
$$

Equation (2-4-15) with (2-4-16) is the expression for the drag on the porous spherical shell. As a special case, the drag on the porous sphere is obtained by letting the inner radius tend to zero, i.e., by letting $\ell_2 \to 0$. In this limit

$$M = 9\ell_1^3(\ell_1\cosh\ell_1 - \sinh\ell_1)$$

$$N = (27\ell_1 + 18\ell_1^3)\cosh\ell_1 - 27\sinh\ell_1$$

and

$$F = 6\pi\mu Ua\left[\frac{2\ell_1^2(1 - \frac{\tanh\ell_1}{\ell_1})}{2\ell_1^3 + 3(1 - \frac{\tanh\ell_1}{\ell_1})}\right]. \tag{2-4-17}$$

Equation (2-4-17) has been derived earlier by Brinkman (1947), Neale *et al.* (1973) and Masliyah *et al.* (1987).

<u>Case II</u> $\sigma = 0$. This case corresponds to when the region III is a solid core of radius b . This problem has been studied, both theoretically and experimentally, by Masliyah *et al.* (1987). In the present case, if we set $\sigma = 0$ in (2-4-9) and (2-4-10)

and again use the definitions of (2-4-12), we find

$$
\begin{aligned}
D &= \frac{4\ell_2^2\ell_1}{C^3\pi}\{-6\ell_2 + (3\ell_2 + 3\ell_1 + \ell_2^3 + 2\ell_1^3)\cosh\triangle + 3(\ell_2^2 - 1)\sinh\triangle\},\\
D_1 &= \frac{12\ell_2^2\ell_1}{C\pi}(\ell_2 - \ell_1\cosh\triangle + \sinh\triangle),\\
D_2 &= -\frac{6\ell_2^2\ell_1}{C^4\pi}\{2\ell_2\ell_1^3 + (3\ell_1\ell_2 + \ell_2^3\ell_1 - 3\ell_2^2)\cosh\triangle\\
&\quad + 3(\ell_2^2\ell_1 - \ell_2^3 - 3\ell_2)\sinh\triangle\},\\
D_3 &= \frac{3\ell_2^2\ell_1}{C^4\pi}\sqrt{\frac{2}{\pi}}\{(\ell_2^3 + 2\ell_1^3)\cosh\ell_2 + 3\ell_2(\cosh\ell_2 - \ell_2\sinh\ell_2\\
&\quad - \cosh\ell_1 + \ell_1\sinh\ell_1)\},\\
D_4 &= -\frac{3\ell_2^2\ell_1}{C^4\pi}\sqrt{\frac{2}{\pi}}\{(\ell_2^3 + 2\ell_1^3)\sinh\ell_2 + 3\ell_2(\sinh\ell_2 - \ell_2\cosh\ell_2\\
&\quad - \sinh\ell_1 + \ell_1\cosh\ell_1)\}.
\end{aligned}
\tag{2-4-18}
$$

Thus the solutions for $\gamma_i\,(i = 1, 2, 3, 4)$ become

$$
\begin{aligned}
\gamma_1 &= 3C^2\{\ell_2 + \sinh\triangle - \ell_1\cosh\triangle\}/J,\\
\gamma_2 &= -\frac{3}{2C}\{2\ell_2\ell_1^3 + (3\ell_2^2\ell_1 - \ell_2^3 - 3\ell_2)\sinh\triangle\\
&\quad + (3\ell_1\ell_2 + \ell_2^3\ell_1 - 3\ell_2^2)\cosh\triangle\}/J,\\
\gamma_3 &= \frac{3}{2C}\sqrt{\frac{\pi}{2}}\{(\ell_2^3 + 2\ell_1^3 + 3\ell_2)\cosh\ell_2\\
&\quad + 3\ell_2(\ell_1\sinh\ell_1 - \cosh\ell_1 - \ell_2\sinh\ell_2)\}/J,\\
\gamma_4 &= -\frac{3}{2C}\sqrt{\frac{\pi}{2}}\{(\ell_2^3 + 2\ell_1^3 + 3\ell_2)\sinh\ell_2\\
&\quad + 3\ell_2(\ell_1\cosh\ell_1 - \sinh\ell_1 - \ell_2\cosh\ell_2)\}/J,\\
J &= -6\ell_2 + (3\ell_2 + 3\ell_1 + \ell_2^3 + 2\ell_1^3)\cosh\triangle + 3(\ell_2^2 - 1)\sinh\triangle.
\end{aligned}
\tag{2-4-19}
$$

Thus, the force on the porous spherical shell with a solid core is given by

$$
F_1 = -4\pi\mu U\alpha_1 = 6\pi\mu Ua\Omega, \tag{2-4-20}
$$

where

$$
\Omega = \frac{1}{\ell_1 J}\{(\ell_2^3 + 3\ell_2 + 2\ell_1^3 - 3\ell_2^2\ell_1)\sinh\triangle + (3\ell_2^2 - \ell_2^3\ell_1 - 3\ell_1\ell_2 - 2\ell_1^4)\cosh\triangle\} \tag{2-4-21}
$$

and J is defined in (2-4-19). We note that the above result agrees with equation (32) of Masliyah *et al.* (1987). Several limiting cases which were discussed in this reference can also be obtained from the present formulae.

A comparison of equation (2-4-20) with (2-4-15) indicates that the force on the porous spherical shell is smaller in magnitude as compared to the shell with the solid core. Masliyah *et al.* (1987) have also shown that the analytical solution for the composite sphere is in excellent agreement with the experimental results. This fact strengthens the validity of the Brinkman equation for these problems and also the usefulness of the solutions given in this work.

5. Asymptotic Expansion and Darcy's Solution.

As Brinkman's equation is compatible with the existence of a boundary layer region at the edge of the porous medium, the asymptotic expansions in the vicinity of the permeable surface based on the exact solutions (2-4-4) to (2-4-6), will now be sought. This will illustrate what the appropriate boundary conditions should be if Darcy's law were to be used within the porous shell. From (2-4-4)-(2-4-7) the following expansions can be derived:

$$
\begin{aligned}
\alpha_1 &= -\frac{3}{2} + \frac{3}{2}\sqrt{k} - \frac{9}{4}\frac{a^2}{(b^3-a^3)}k + O(k^{\frac{3}{2}}),\\
\alpha_2 &= \frac{3}{4}a^3 - \frac{9}{4}a^2\sqrt{k} - \frac{9}{8}\frac{a^4}{(b^3-a^3)}k + O(k^{\frac{3}{2}}),\\
\beta_1 &= -\frac{45k}{(b^3-a^3)b^3}\{-ba + (b+5a)\sqrt{k} + O(k)\},\\
\beta_2 &= \frac{k}{(b^3-a^3)}\{-9a + (9 + \frac{a}{b}\frac{45}{2})\sqrt{k} + O(k)\},
\end{aligned}
\qquad (2\text{-}5\text{-}1)
$$

and then the velocities may be written as

$$
\begin{aligned}
u_r^{(1)} &= U\cos\theta\{(1+\frac{1}{2}\frac{a^3}{r^3}-\frac{3}{2}\frac{a}{r})+(-\frac{3}{2}\frac{a^2}{r^3}+\frac{3}{2}\frac{1}{r})\sqrt{k} \\
&\quad -\frac{3}{4}\frac{1}{(b^3-a^3)}(\frac{a^4}{r^3}+3\frac{a^2}{r})k+O(k^{\frac{3}{2}})\}, \qquad (2\text{-}5\text{-}2) \\
u_\theta^{(1)} &= -U\sin\theta\{(1-\frac{1}{4}\frac{a^3}{r^3}-\frac{3}{4}\frac{a}{r})+(\frac{3}{4}\frac{a^2}{r^3}+\frac{3}{4}\frac{1}{r})\sqrt{k} \\
&\quad -\frac{3}{8}\frac{1}{(b^3-a^3)}(-\frac{a^4}{r^3}+3\frac{a^2}{r})k+O(k^{\frac{3}{2}})\}, \qquad (2\text{-}5\text{-}3) \\
u_r^{(3)} &= U\cos\theta(\beta_2+\frac{1}{10}\beta_1 r^2) \\
&= U\cos\theta\frac{k}{(b^3-a^3)}\{(-9a+\frac{9r^2}{2b^2}a) \\
&\quad +[(9+\frac{a}{b}\frac{45}{2})-\frac{9}{2}(b+5a)\frac{r^2}{b^3}]\sqrt{k}+O(k)\}, \qquad (2\text{-}5\text{-}4) \\
u_\theta^{(3)} &= -U\sin\theta(\beta_2+\frac{1}{5}\beta_1 r^2) \\
&= -U\sin\theta\frac{k}{(b^3-a^3)}\{(-9a+\frac{9r^2}{b^2}a) \\
&\quad +[(9+\frac{a}{b}\frac{45}{2})-9(b+5a)\frac{r^2}{b^3}]\sqrt{k}+O(k)\}. \qquad (2\text{-}5\text{-}5)
\end{aligned}
$$

The solution of Brinkman's equation in the porous spherical shell has the following form:

$$v_r = U\cos\theta k\{-\gamma_1+2\gamma_2 r^{-3}+2\gamma_3 y_2(r)+2\gamma_4 y_2^*(r)\}, \qquad (2\text{-}5\text{-}6)$$

$$v_\theta = U\sin\theta k\{\gamma_1+\gamma_2 r^{-3}+\gamma_3(y_2(r)-C^2 y_1(r))+\gamma_4(y_2^*(r)-C^2 y_1^*(r))\}. \qquad (2\text{-}5\text{-}7)$$

As can be seen from the above the expressions these velocities are composed of two parts. The first part contains the first two terms which are recognized as Darcy's solution. The second part is extremely small outside the boundary layers and rapidly changes in the boundary layers.

In other words, this part, which is determined by the viscous term in the

Brinkman equation, manifests the existence of the boundary layers at the two permeable surfaces, and cannot be obtained through Darcy's solution.

We also have

$$\gamma_1 = \frac{3}{2}\frac{1}{(b^3-a^3)}\{a - \sqrt{k} + O(k)\},$$
$$\gamma_2 = -\frac{3}{2}\frac{1}{(b^3-a^3)}\{ab^3 - b^3\sqrt{k} + O(k)\}, \tag{2-5-8}$$

Consequently, outside of the boundary layers, (2-5-6) and (2-5-7) have expansions

$$v_r = \frac{3}{2}U\cos\theta\frac{k}{(b^3-a^3)}\{-(a+\frac{2ab^3}{r^3}) + (1+\frac{2b^3}{r^3})\sqrt{k} + O(k)\}, \tag{2-5-9}$$

$$v_\theta = \frac{3}{2}U\sin\theta\frac{k}{(b^3-a^3)}\{(a-\frac{ab^3}{r^2})(-1+\frac{b^3}{r^3})\sqrt{k} + O(k)\},$$
$$b+\delta \le r \le a-\delta, \tag{2-5-10}$$

where δ is the thickness of the boundary layers.

If q_r and q_θ are normal and tangential components of the velocities described by Darcy's law within the porous shell, they will be indistinguishable from (2-5-9) and (2-5-10) outside of the boundary layers. This indicates that q_r and q_θ have the same expansions as (2-5-9) and (2-5-10) within the whole porous shell.

Based on the above expansions, we are able to propose the boundary conditions to be applied at the permeable interfaces when Darcy's law is used in a porous medium.

(a) At the outer surface of the porous shell, we have

$$u_\theta^{(1)} = \sqrt{k}\frac{\partial u_\theta^{(1)}}{\partial r} + O(k) \qquad \text{at } r=a, \tag{2-5-11}$$

$$e_{r\theta} = r\frac{\partial}{\partial r}(\frac{u_\theta^{(1)}}{r}) + \frac{1}{r}\frac{\partial u_r^{(1)}}{\partial\theta}$$
$$= \frac{1}{\sqrt{k}}u_\theta^{(1)} + O(\sqrt{k}) \qquad \text{at } r=a, \tag{2-5-12}$$

or

$$e_{r\theta} = \frac{1}{\sqrt{k}}(u_\theta^{(1)} - q_\theta) + O(\sqrt{k}) \qquad \text{at } r = a, \tag{2-5-13}$$

The condition (2-5-11) is Saffman's condition when Saffman's λ-parameter is set equal to one (Saffman 1971).

The condition (2-5-12) or (2-5-13) is Beaver and Joseph's condition, also with the empirical parameter λ equal to one (Beaver and Joseph 1967).

Both conditions are equivalent at the surface, as the same order approximation appears in both cases.

We also note that

$$u_r^{(1)} = O(k) \qquad \text{at} \qquad r = a, \tag{2-5-14}$$

which can be rewritten as

$$u_r^{(1)} = q_r + O(k) \qquad \text{at} \qquad r = a, \tag{2-5-15}$$

considering $q_r \sim o(k)$ at $r = a$.

(b) At the inner surface of the porous shell, we have

$$\begin{aligned} e_{r\theta} &= \frac{\partial u_\theta^{(3)}}{\partial r} - \frac{u_\theta^{(3)}}{r} + \frac{1}{r}\frac{\partial u_r^{(3)}}{\partial r} \\ &= -\frac{1}{\sqrt{k}} u_\theta^{(3)} + O(\sqrt{k}) \qquad \text{at } r = b, \end{aligned} \tag{2-5-16}$$

or

$$e_{r\theta} = -\frac{1}{\sqrt{k}}(u_\theta^{(3)} - q_\theta) + O(\sqrt{k}), \qquad \text{at } r = b \tag{2-5-17}$$

and

$$u_r^{(3)} = q_r + O(k) \qquad \text{at } r = b. \tag{2-5-18}$$

We notice here that only Beaver and Joseph's condition can hold and Saffman's condition is no longer equivalent to Beaver and Joseph's condition.

(c) Jones (1973) has discussed the problem of viscous flow around a porous spherical shell by using Darcy's law for a porous medium and applying the following boundary conditions.

i) Continuity of pressure,

ii) $u_r = q_r$,

iii)
$$e_{r\theta} = \beta(u_\theta - q_\theta) \quad \text{at} \quad r = a,$$
$$e_{r\theta} = -\beta(u_\theta - q_\theta) \quad \text{at} \quad r = b, \tag{2-5-19}$$

where $\beta = \frac{\alpha}{\sqrt{k}}$.

The conditions stated in (a) and (b) above agree with these conditions.

(d) As a comparison, the case $\sigma = 0$ should be mentioned. The corresponding problem was discussed in detail by Haber and Mauri (1983).

In this case, at the outer surface the following relations still hold

$$u_\theta^{(1)} = \sqrt{k}\frac{\partial u_\theta^{(1)}}{\partial r} + O(k) \qquad \text{at} \quad r = a,$$
$$e_{r\theta} = \frac{1}{\sqrt{k}}u_\theta^{(1)} + O(\sqrt{k}) \qquad \text{at} \quad r = a. \tag{2-5-20}$$

but at the inner surface (impermeable surface) a slip condition must be applied for normal velocity, i.e.,

$$q_r = \sqrt{k}\frac{\partial q_r}{\partial r} \qquad \text{at} \quad r = b$$

if Darcy's law is used in the porous media.

We note that in this case, we have

$$\begin{aligned} v_r &= U\cos\theta\frac{3a^3k}{2a^3+b^3}\{(1-\frac{b^3}{r^3}) + [-3\frac{ab^3}{r^3} + (\frac{b^3}{r^3}-1) \\ &\quad (\frac{3ab^2}{2a^3+b^3}+1)]\sqrt{k} + o(k)\}, \\ v_\theta &= \frac{1}{2}U\sin\theta\frac{3a^3k}{2a^3+b^3}\{(2+\frac{b^3}{r^3}) + [3\frac{ab^2}{r^3} - (\frac{b^3}{r^3}+2) \\ &\quad (\frac{3ab^2}{2a^3+b^3}+1)]\sqrt{k} + O(k)\}. \end{aligned} \tag{2-5-21}$$

Finally, we remark here that more applications are available using the exact solution in the closed form obtained in section 1, for example, in the problem of quadratic flow past a porous sphere (cf. Kaloni and Qin 1992).

6. Conclusions.

In this chapter we have developed a Cartesian-tensor solution of the Brinkman equation. The use and the simplicity of the method has been illustrated by two examples. In the first example the flow past a porous sphere is considered and it is found that the drag on a porous sphere is lower than that of a solid sphere. It has also been shown that the drag on the sphere decreases as the permeability of the porous medium increases. The second example deals with creeping flow past a spherical shell. After presenting the general solution for the situation, when the viscosities in all the three regions can be different, several limiting cases have been discussed. In the final section an asymptotic expansion of the exact solution of the Brinkman equation in the vicinity of the permeable surface is carried out. It is shown that Beaver and Joseph's or Saffman's slip boundary condition is recovered from our results with the specific choice of the values for the empirical parameters.

Chapter 3

Steady Convection in a Porous Medium

1. Introduction.

Many problems in fluid dynamics are reduced to solving boundary value problems for partial differential equations. In many cases, it is impossible to get the explicit form of solutions. Before any attempt is made to solve these problems approximately or numerically, a mathematical concern about whether the problems are well-posed or not is necessary. Roughly speaking, a problem is said to be well-posed if a unique solution exists which depends continuously on the initial or boundary data. To make the statement precise we must indicate in what space the solution is to lie , and point out some measure of the continuous dependence. Here we take a theoretical look, from this perspective, at steady convection problems in a porous medium.

In this chapter we employ the Brinkman model to discuss the existence of weak solutions, via variational formulation, for a steady convection flow problem in a porous medium. Some regularity properties and uniqueness of solutions are also studied. We point out that Polisěvski (1985 a,b) has studied the weak continuity and regularity of solutions in convective problems using Darcy's law. The method employed in this chapter is similar to the method used in dealing with the Navier-Stokes equations, as summarized in the book of Temam (1977). The addition of the Laplacian term of Brinkman's equation to the velocity field in Darcy's law allows us to use Galerkin's method in a Hilbert space to deal with the existence of solutions, and to use some deep results of elliptic partial differential equations to study other properties of solutions. Our approach to the existence of solutions is also amenable to numerical approximation.

Let Ω be an open connected bounded domain in R^n ($n = 2$, or 3) with a boundary $\partial\Omega$ of class C^2. We consider the problem of steady convection flow in a fluid-saturated porous medium. Under the assumption that the porous medium is in local thermal equilibrium, the Boussinesq approximation is applicable. As a result, the governing equations with inertial and thermal dispersion effects neglected can be written as (cf. Hsu and Cheng 1985)

$$div\, \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{3-1-1}$$

$$-\frac{\mu}{\phi}\nabla^2\mathbf{u} + \mu\mathbf{M}\mathbf{u} + \nabla\bar{p} = \rho_0[1 - \beta(T - T_0)]\mathbf{g} \qquad \text{in } \Omega, \tag{3-1-2}$$

$$-div\,(\mathbf{N}\nabla T) + \mathbf{u}\cdot\nabla T = 0 \qquad \text{in } \Omega, \tag{3-1-3}$$

where $\mathbf{u}$ is the volume averaged macroscopic velocity and $\bar{p}$ is an intrinsic average pressure. Both $\mathbf{u}$ and $\bar{p}$ are physically measurable quantities. The symbols ρ_0, β and μ are the mean density of the fluid, the thermal expansion coefficient of the fluid and the viscosity of the fluid, respectively. $\mathbf{g}$ is the potential type gravitational acceleration, $\mathbf{M}^{-1} = \mathbf{k}$ is the positive symmetric constant tensor of permeability, $\mathbf{N}$ is the positive constant tensor of thermal diffusion, ϕ is the porosity, T is the temperature and $T_0 > 0$ is a uniform reference temperature. The boundary conditions are:

$$\mathbf{u} = \mathbf{a} \qquad \text{on } \partial\Omega, \tag{3-1-4}$$

$$T = \zeta \qquad \text{on } \partial\Omega. \tag{3-1-5}$$

We will suppose that the field $\mathbf{a}(x)$ on $\partial\Omega$ can be extended inside the domain Ω in the form $\mathbf{a}(x) = \operatorname{curl}\mathbf{b}(x)$; $\zeta(x)$ is the nonuniform temperature which also can be extended inside the domain Ω. The smoothness conditions imposed on $\mathbf{a}(x)$ and $\zeta(x)$ will be specified later in the following sections. In this section we assume $a_i(x)$, $i = 1, \dots, n$ and $\zeta(x)$ are functions in $C^2(\Omega) \cap C^0(\bar{\Omega})$. A uniform reference temperature T_0 could be chosen as $T_0 = \frac{1}{2}(\sup_{x\in\partial\Omega}\zeta + \inf_{x\in\partial\Omega}\zeta)$ for convenience. The solenoidally extended function $\mathbf{a}(x)$ satisfies $div\,\mathbf{u} = 0$.

In order to obtain homogeneous boundary conditions, we set

$$\hat{\mathbf{u}} = \mathbf{u} - \mathbf{a}, \qquad S = T - (\psi + T_0) \tag{3-1-6}$$

and write $p = \bar{p} - \sum_{i=1}^{n} \rho_0 g_i x_i$, where $\psi = \zeta - T_0$ and p is called hydrostatic pressure. The problem (3-1-1)-(3-1-5) then becomes

$$div\,\hat{\mathbf{u}} = 0 \qquad \text{in } \Omega, \tag{3-1-7}$$

$$-\frac{\mu}{\phi}\Delta^2(\hat{\mathbf{u}} + \mathbf{a}) + \mu\mathbf{M}(\hat{\mathbf{u}} + \mathbf{a}) + \nabla p + \rho_0\beta(S + \psi)\mathbf{g} = 0 \qquad \text{in } \Omega, \tag{3-1-8}$$

$$-div\,(\mathbf{N}\nabla(S + \psi)) + (\hat{\mathbf{u}} - \mathbf{a})\cdot\nabla(S + \psi) = 0 \qquad \text{in } \Omega, \tag{3-1-9}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega, \tag{3-1-10}$$

$$S = 0 \qquad \text{on } \partial\Omega. \tag{3-1-11}$$

We now introduce the following dimensionless variables:

$$\mathbf{x}^* = L^{-1}\mathbf{x}; \qquad \mathbf{x}^* \in \Omega^* \longleftrightarrow \mathbf{x} \in \Omega;$$

$$\mathbf{u}^* = (\beta\rho_0\xi g)^{-1}\mu m_1 \hat{\mathbf{u}}; \quad p^* = (\beta\rho_0\xi g L)^{-1}p; \quad \mathbf{a}^* = (\beta\rho_0\xi g)^{-1}\mu m_1 \mathbf{a};$$

$$S^* = \xi^{-1}S; \qquad \psi^* = \xi^{-1}\psi; \qquad T_0^* = \xi^{-1}T_0;$$

$$\mathbf{M}^* = m_1^{-1}\mathbf{M}; \qquad N^* = n_1^{-1}\mathbf{N}; \qquad \mathbf{g}^* = g^{-1}\mathbf{g},$$

where L is the edge length of the $n-$ cube in which Ω can be included, $\xi = \frac{1}{2}(\sup_{x\in\partial\Omega}\zeta - \inf_{x\in\partial\Omega}\zeta)$, $g = |\mathbf{g}|$, and m_1, n_1 are respectively the smallest eigenvalues of M and N. Thus the smallest eigenvalues of M^* and N^* are 1, and Ω^* is included in a n-cube of edge length 1.

Omitting the asterisk *, we obtain a nondimensional form of (3-1-7)-(3-1-11) as:

$$div\,\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{3-1-12}$$

$$-\sigma\nabla^2(\mathbf{u}+\mathbf{a}) + \mathbf{M}(\mathbf{u}+\mathbf{a}) + \nabla p + (S+\psi)\mathbf{g} = 0 \qquad \text{in } \Omega, \tag{3-1-13}$$

$$-div\,(\mathbf{N}\nabla(S+\psi)) + R(\mathbf{u}+\mathbf{a})\cdot\nabla(S+\psi) = 0 \qquad \text{in } \Omega, \tag{3-1-14}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega, \tag{3-1-15}$$

$$S = 0 \qquad \text{on } \partial\Omega, \tag{3-1-16}$$

where R is the nondimensional Rayleigh number defined by

$$R = \beta\rho_0\xi g(\mu m_1 n_1)^{-1} \tag{3-1-17}$$

and

$$\sigma = D_a/\phi, \tag{3-1-18}$$

where $D_a = \frac{1}{m_1 L^2}$ is the Darcy number.

2. Weak Formulation.

We shall use the definitions and notations for the scalar products and norms given in Chapter 1. Let $\mathbf{L}^2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ be two spaces defined in (1-14)-(1-16). The closure of $\mathcal{V}$ in $\mathbf{L}^2(\Omega)$ and in $\mathbf{H}_0^1(\Omega)$ are two basic spaces in the study of the present problem. The characterizations of those two spaces are

$$\mathbf{H} = \{\, \mathbf{u} \in \mathbf{L}^2(\Omega) \mid div\,\mathbf{u} = 0 \text{ in } \Omega,\ \ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \,\}, \tag{3-2-1}$$

where $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}$ should be understood as $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = \lim_{m\to\infty}\mathbf{u}_m\cdot\mathbf{n}|_{\partial\Omega} = 0$, if $\mathbf{u} = \lim_{m\to\infty}\mathbf{u}_m$ in $\mathbf{L}^2(\Omega)$ for $\mathbf{u}_m \in \mathcal{V}$.

$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) \mid div\,\mathbf{u} = 0 \ \text{ in } \Omega\}. \tag{3-2-2}$$

We also suppose that the product Hilbert space $\mathbf{V} \times H_0^1(\Omega)$ is equipped with usual scalar product

$$((\mathbf{u}, S), (\mathbf{v}, t)) = ((\mathbf{u}, \mathbf{v}))_{\mathbf{H}_0^1(\Omega)} + (S, t)_{H_0^1(\Omega)}. \tag{3-2-3}$$

Let us assume that $\mathbf{u}$, p and S are smooth functions satisfying equations (3-1-12)-(3-1-16) in the ordinary sense, and $\mathbf{a}$ and ψ are known functions which are also assumed as smooth functions. Taking scalar products of (3-1-13) and (3-1-14) with the functions $\mathbf{V} \in \mathcal{V}$, $t \in \mathfrak{D}(\Omega)$, respectively, integrating by parts and using the equations (3-1-15) and (3-1-16), we obtain

$$\sigma(\nabla(\mathbf{u}+\mathbf{a}), \nabla\mathbf{v})_{\mathbf{L}^2(\Omega)} + (\mathbf{M}(\mathbf{u}+\mathbf{a}), \mathbf{v})_{\mathbf{L}^2(\Omega)} + (S+\psi, \mathbf{g}\cdot\mathbf{v})_{L^2(\Omega)} = 0, \tag{3-2-4}$$

$$(\mathbf{N}\nabla(S+\psi), \nabla t)_{\mathbf{L}^2(\Omega)} + R((\mathbf{u}+\mathbf{a})\cdot\nabla(S+\psi), t)_{L^2(\Omega)} = 0. \tag{3-2-5}$$

From now on we shall omit subscript $L^2(\Omega)$ in the scalar product for simplicity. Since space V is the closure of $\mathcal{V}$ in $\mathbf{H}_0^1(\Omega)$, $\mathfrak{D}(\Omega)$ is a dense subset in $H_0^1(\Omega)$, a continuity argument shows that (3-2-4) and (3-2-5) are still satisfied if $(\mathbf{u}, S) \in \mathbf{V} \times H_0^1(\Omega)$ for any $(\mathbf{v}, t) \in \mathbf{V} \times H_0^1(\Omega)$. Let us define a mapping $G(\cdot,\cdot)$ from $\mathbf{V} \times H_0^1(\Omega)$ into itself by

$$\begin{aligned} < G(\mathbf{u}, S), (\mathbf{v}, t) > &= \sigma(\nabla(\mathbf{u}+\mathbf{a}), \nabla\mathbf{v}) + (\mathbf{M}(\mathbf{u}+\mathbf{a}), \mathbf{v}) \\ &+ (S+\psi, \mathbf{g}\cdot\mathbf{v}) + (\mathbf{N}\nabla(S+\psi), \nabla t) + R((\mathbf{u}+\mathbf{a})\cdot\nabla(S+\psi), t) \\ &\qquad \forall(\mathbf{v}, t) \in \mathbf{V} \times H_0^1(\Omega). \end{aligned} \tag{3-2-6}$$

Thus the weak formulation associated with problem (3-1-12)-(3-1-16) becomes:

To find a $(\mathbf{u}, S) \in \mathbf{V} \times H_0^1(\Omega)$ such that

$$< G(\mathbf{u}, S), (\mathbf{v}, t) > = 0 \qquad \forall(\mathbf{v}, t) \in \mathbf{V} \times H_0^1(\Omega). \tag{3-2-7}$$

Conversely if $(\mathbf{u}, S) \in \mathbf{V} \times H_0^1(\Omega)$ satisfies (3-2-7) then (3-2-4) and (3-2-5) hold for any $v \in \mathbf{V}$, and $t \in H_0^1(\Omega)$ by choosing $t = 0$ or $\mathbf{v} = 0$ in (3-2-7), respectively. By virtue of propositions 1.1 and 1.2 in the book of Temam (1977 Chap. I), the following results are true:

Let Ω be an open set in R^n and $f = \{f_1, \dots, f_n\}$, $f_i \in \mathcal{D}'(\Omega)$, $i = 1, \dots, n$.

(i) A necessary and sufficient condition that $\mathbf{f} = \operatorname{grad} p$ for some p in $\mathcal{D}'(\Omega)$, is that

$$< \mathbf{f}, \mathbf{v} >= 0 \qquad \forall \mathbf{v} \in \mathcal{V}.$$

(ii) Let Ω be a bounded Lipschitz open set in R^n. If a distribution p has all its first derivatives $D_i p$, $1 \le i \le n$ in $H^{-1}(\Omega)$, then $p \in L^2(\Omega)$.

It follows from (3-2-4) that there exists a distribution $p \in L^2(\Omega)$ such that (3-1-13) is true in the distribution sense in Ω. Also (3-2-5) implies that (3-1-14) holds in the distribution sense in Ω and (3-1-12),(3-1-15) and (3-1-16) are satisfied in the distribution sense in Ω and in the trace sense on $\partial\Omega$, respectively.

3. The Existence of Solutions.

To discuss the existence of weak solutions we assume the extended boundary functions $a_i(x) \in H^2(\Omega)$, $i = 1, \dots, n$, $\zeta(x) \in H^2(x)$. The restrictions of them on the boundary, therefore, are continuous functions so that $T_0 = \frac{1}{2}(\sup_{x\in\partial\Omega} \zeta + \inf_{x\in\partial\Omega} \zeta)$ makes sense and $\int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n}\, dx = 0$ holds.

For later discussion we will construct a function ψ_δ depending on a $\delta > 0$. The method used here was introduced half a century ago by E. Hopf (1940). Let $\rho(x) = d(x, \Gamma) =$ the shortest distance from x to Γ, $\Gamma = \partial\Omega$. For any $\epsilon > 0$, because Γ is of class C^2, there exists a function $\theta_\epsilon \in C^2(\bar{\Omega})$ such that

(i) $\theta_\epsilon = 1$ in some neighbourhood of Γ (which depends on ϵ),

(ii) $\theta_\epsilon = 0$ if $\rho(x) \ge 2\delta(\epsilon)$, where $\delta(\epsilon) = exp(-\frac{1}{\epsilon})$,

(iii) $|D_k\theta_\epsilon| \le \epsilon/\rho(x)$ if $\rho(x) < 2\delta(\epsilon)$, $k = 1,\dots,n$ and $D_k = \partial/\partial x_k$.

Let $\psi_\epsilon = (\zeta - T_0)\theta_\epsilon$. It is obvious that $\psi_\epsilon \in H^2(\Omega)$, and for any $S \in H_0^1(\Omega)$ we have

$$\begin{aligned}
\|S\nabla\psi_\epsilon\|_{L^2(\Omega)} &= (\sum_{k=1}^{n}\int_\Omega (SD_k\psi_\epsilon)^2\,dx)^{1/2} \\
&\le \{2\sum_{k=1}^{n}\int_\Omega [S^2(\zeta - T_0)^2(D_k\theta_\epsilon)^2 + S^2\theta_\epsilon^2(D_k\zeta)^2]\,dx\}^{1/2} \\
&\le \epsilon\sqrt{2n}\,\|\zeta - T_0\|_{L^\infty(\Omega)}\left\|\frac{S}{\rho}\right\|_{L^2(\Omega)} + \eta(\epsilon)\|S\|_{L^4(\Omega)},
\end{aligned} \tag{3-3-1}$$

where $\eta(\epsilon) = \sqrt{2}\,\{\sum_{k=1}^{n}\int_{\rho(x)\le 2\delta(\epsilon)}(D_k\zeta)^4\,dx\}^{1/4} \to 0$, if $\epsilon \to 0$ since $\zeta \in H^2(\Omega)$.

With the useful Hardy's inequality (see Hopf p.768)

$$\left\|\frac{S}{\rho}\right\|_{L^2(\Omega)} \le \text{const.}\,\|S\|_{H_0^1(\Omega)} \qquad \forall S \in H_0^1(\Omega)$$

and the Sobolev's inequality

$$\|S\|_{L^4(\Omega)} \le \text{const.}\|S\|_{H_0^1(\Omega)} \qquad \forall S \in H_0^1(\Omega),$$

the relationship (3-3-1) becomes

$$\|S\nabla\psi_\epsilon\|_{L^2(\Omega)} \le \text{const.}\max(\epsilon, \eta(\epsilon))\,\|S\|_{H_0^1(\Omega)} \qquad \forall S \in H_0^1(\Omega).$$

Thus, for any $\delta > 0$ we can choose a sufficiently small ϵ such that

$$\|S\nabla\psi_\delta\|_{L^2(\Omega)} \le \delta\,\|S\|_{H_0^1(\Omega)}, \qquad \forall S \in H_0^1(\Omega). \tag{3-3-2}$$

Because at the boundary ψ_δ and ψ are coincident, from now on we use ψ_δ instead of ψ as the known boundary data. If we write ψ_δ instead of ψ in (3-2-6), the following result holds

Lemma 3.1. *If G is a mapping from $\mathbf{V} \times H_0^1(\Omega)$ into itself defined by (3-2-6) then*

(i) G is continuous,

(ii) there exists an $r > 0$, such that

$$< G(\xi), \xi >> 0, \quad \forall \xi \in \mathbf{V} \times H_0^1(\Omega) \ \ \textit{with} \ \|\xi\|_{\mathbf{V}\times H_0^1(\Omega)} = r.$$

<u>Proof.</u> Let $(\mathbf{u}^k, S^k) \to (\mathbf{u}, S)$ strongly in $\mathbf{V} \times H_0^1(\Omega)$ as $k \to \infty$, m_l and n_l be the largest eigenvalues of matrices $\mathbf{M}$ and $\mathbf{N}$. For any $(\mathbf{v}, t) \in \mathbf{V} \times H_0^1(\Omega)$, we have

$$\begin{aligned}
&| < G(\mathbf{u}^k, S^k) - G(\mathbf{u}, S), (\mathbf{v}, t) > | \\
&\leq \sum_{i,j=1}^{n} \{\sigma \int_\Omega |D_j(u_i^k - u_i) D_j v_i| \, dx \\
&\qquad + \int_\Omega |M_{i,j}(u_i^k - u_i) v_j| \, dx + \int_\Omega |N_{ij} D_i (S^k - S) D_j t| \, dx\} \\
&\qquad + \sum_{i=1}^{n} \{ \int_\Omega |(S^k - S) g_i v_i| \, dx + R \int_\Omega |(u_i^k + a_i)(S^k - S) D_i t| \, dx\} \\
&\qquad + R \int_\Omega |(u_i^k - u_i)(S + \psi_\delta) D_i t| \, dx\} \\
&\leq \sigma \|\mathbf{u}^k - \mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}} + m_l \|\mathbf{u}^k - \mathbf{u}\|_{\mathbf{H}} \|\mathbf{v}\|_{\mathbf{H}} \\
&\qquad + n_l \|S^k - S\|_{H_0^1(\Omega)} \|t\|_{H_0^1(\Omega)} + \|S_k - S\|_{L^2(\Omega)} \|\mathbf{v}\|_{\mathbf{H}} \\
&\qquad + n^{\frac{1}{4}} R \|t\|_{H_0^1(\Omega)} \|\mathbf{u}^k + \mathbf{a}\|_{\mathbf{L}^4(\Omega)} \|S^k - S\|_{L^4(\Omega)} \\
&\qquad + n^{\frac{1}{4}} R \|\mathbf{u}^k - \mathbf{u}\|_{\mathbf{L}^4(\Omega)} \|S + \psi_\delta\|_{L^4(\Omega)} \|t\|_{H_0^1(\Omega)},
\end{aligned} \tag{3-3-3}$$

where the Hölder's inequality has been used. The continuity of G follows from (3-3-3) and the Sobolev's imbedding theorems (see Gilbarg and Trudinger 1983).

Now we prove the second part of Lemma 3.1. For any $(\mathbf{u}, S) \in \mathbf{V} \times H_0^1(\Omega)$, we have

$$
\begin{aligned}
< G(\mathbf{u}, S), (\mathbf{u}, S) > \geq\ & \sigma\|\mathbf{u}\|_{\mathbf{V}}^2 + \|\mathbf{u}\|_{\mathbf{H}}^2 + \|S\|_{H_0^1(\Omega)}^2 \\
& - \sigma\|\nabla \mathbf{a}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}\|_{\mathbf{V}} - m_l \|\mathbf{a}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}\|_{\mathbf{H}} \\
& - \|S\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{H}} - \|\psi_\delta\|_{L^2(\Omega)} \|\mathbf{u}\|_{\mathbf{H}} - n_l \|\nabla \psi_\delta\|_{\mathbf{L}^2(\Omega)} \|S\|_{H_0^1(\Omega)} \\
& - \delta R(\|\mathbf{u}\|_{\mathbf{H}} + \|\mathbf{a}\|_{\mathbf{L}^2(\Omega)}) \|S\|_{H_0^1(\Omega)},
\end{aligned} \tag{3-3-4}
$$

where the last term in (3-3-4) was obtained by using (3-3-2).

Recalling the well-known inequalities (see Ladyzhenskaja 1969)

$$
\begin{aligned}
\|S\|_{L^2(\Omega)} &\leq \frac{1}{\pi\sqrt{n}} \|S\|_{H_0^1(\Omega)} && \forall S \in H_0^1(\Omega), \\
\|\mathbf{u}\|_{\mathbf{H}} &\leq \frac{1}{\pi\sqrt{n}} \|\mathbf{u}\|_{\mathbf{V}} && \forall \mathbf{u} \in \mathbf{V},
\end{aligned}
$$

and noting

$$
\|\mathbf{u}\|_{\mathbf{H}} \|S\|_{L^2(\Omega)} \leq \|\mathbf{u}\|_{\mathbf{H}}^2 + \frac{1}{4}\|S\|_{L^2(\Omega)}^2,
$$

we have

$$
\begin{aligned}
< G(\mathbf{u}, S), (\mathbf{u}, S) > \geq\ & (\sigma - \frac{R\delta}{2\pi\sqrt{n}}) \|\mathbf{u}\|_{\mathbf{V}}^2 \\
& + (1 - \frac{1}{4\pi\sqrt{n}} - \frac{R\delta}{2\pi\sqrt{n}}) \|S\|_{H_0^1(\Omega)}^2 \\
& - n_l \|\psi_\delta\|_{H^1(\Omega)} (\|\mathbf{u}\|_{\mathbf{V}}^2 + \|S\|_{H_0^1(\Omega)}^2)^{1/2} \\
& - (\sigma + \frac{m_l}{\pi\sqrt{n}} + R\delta) \|\mathbf{a}\|_{\mathbf{H}^1(\Omega)} (\|\mathbf{u}\|_{\mathbf{V}}^2 + \|S\|_{H_0^1(\Omega)}^2)^{1/2} \\
\geq\ & (\gamma - \frac{R\delta}{2\pi\sqrt{n}}) \|(\mathbf{u}, S)\|_{\mathbf{V}\times H_0^1(\Omega)}^2 \\
& - [\, n_l \|\psi_\delta\|_{H^1(\Omega)} + (\sigma + \frac{m_l}{\pi\sqrt{n}} + R\delta) \|\mathbf{a}\|_{\mathbf{H}^1(\Omega)} \,] \|(\mathbf{u}, S)\|_{\mathbf{V}\times H_0^1(\Omega)},
\end{aligned} \tag{3-3-5}
$$

where

$$
\gamma = \min (\sigma, \frac{4\pi\sqrt{n} - 1}{4\pi\sqrt{n}})
$$

and

$$\|\psi_\delta\|_{H^1(\Omega)} = (\,\|\psi_\delta\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \|D_i\psi_\delta\|^2_{L^2(\Omega)}\,)^{1/2},$$

$$\|\mathbf{a}\|_{\mathbf{H}^1(\Omega)} = (\,\|\mathbf{a}\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^{n} \|D_i\mathbf{a}\|^2_{\mathbf{L}^2(\Omega)}\,)^{1/2}.$$

The lemma is proved by choosing

$$\delta < \frac{2\pi\sqrt{n}\gamma}{R_a}$$

and

$$r > \frac{n_l\|\psi_\delta\|_{H^1(\Omega)} + (\sigma + \frac{m_l}{\pi\sqrt{n}} + R\delta)\,\|\mathbf{a}\|_{H^1(\Omega)}}{\gamma - \frac{R}{2\pi\sqrt{n}}}.$$

In order to obtain the result of existence we need the next lemma (see Lions 1969: p. 53).

Lemma 3.2. *Let X be a finite-dimensional Hilbert space with scalar product$(\cdot,\cdot)$ and norm $\|\cdot\|$, and let G be a continuous mapping from X into itself such that*

$$(\,G(\xi), \xi\,) > 0 \qquad \textit{for} \ \ \|\xi\| = k > 0.$$

Then there exists $\xi \in X$, with $\|\xi\| \le k$, such that

$$G(\xi) = 0$$

We are now in the position to obtain the main result of this section.

Theorem 3.3. *The problem (3-2-7) has at least one solution $(\mathbf{u}, S) \in \mathbf{V} \times H^1_0(\Omega)$.*

Proof. The existence of solution ($\mathbf{u}$, S) is to be proved by the Galerkin method, that is, we construct an approximate solution and then pass to the appropriate limits.

The space $\mathbf{V}$ is separable as a subspace of $\mathbf{H}_0^1(\Omega)$ which is separable. There exist two sequences $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_m$ of linearly independent elements in $\mathbf{V}$ and $R_1, R_2, \dots, R_m$ of linearly independent elements in $H_0^1(\Omega)$. We would like to define an approximate solution $(\mathbf{u}_m, S_m)$ of (3-2-7) by

$$\mathbf{u}_m = \sum_{k=1}^{m} \xi_k^m \mathbf{W}_k, \qquad S_m = \sum_{l=1}^{m} \eta_l^m R_l, \tag{3-3-6}$$

$$\sigma(\nabla(\mathbf{u}_m + \mathbf{a}), \nabla\mathbf{W}_k) + (\mathbf{M}(\mathbf{u}_m + \mathbf{a}), \mathbf{W}_k) + ((S_m + \psi_\delta), \mathbf{g}\cdot\mathbf{W}_k) = 0, \tag{3-3-7}$$

$$(\mathbf{N}\nabla(S + \psi_\delta),\, \nabla R_l) + R((\mathbf{u}_m + \mathbf{a})\cdot\nabla(S_m + \psi_\delta),\, R_l) = 0, \tag{3-3-8}$$

$$\xi_k^m,\ \eta_l^n \in R, \qquad k, l = 1, \dots, m.$$

The equations (3-3-6)-(3-3-8) are a system of nonlinear equations for the constant coefficients ξ_k^m and η_l^m. The existence of a solution of this system will be obtained from the following argument.

Let X be the product space spanned by $\mathbf{W}_1, \dots, \mathbf{W}_m$ and $R_1, \dots, R_m$. The scalar product on X is induced by $\mathbf{V} \times H_0^1(\Omega)$, and $G = G_m$ is defined by

$$< G_m(\mathbf{u}, S), (\mathbf{v}, t) >=< G(\mathbf{u}, S), (\mathbf{v}, t) >= \text{(3-2-6)} \qquad \forall(\mathbf{u}, S), (\mathbf{v}, t) \in X.$$

As the topological structure of X is the same as that of $\mathbf{V} \times H_0^1(\Omega)$, it is obvious that G_m satisfies the hypothesis of Lemma 3.2, which we can verify by following the same procedure as in the proof of Lemma 3.1. Therefore, there exists a solution $(\mathbf{u}_m, S_m) \in X$ such that

$$< G_m(\mathbf{u}_m, S_m), (\mathbf{v}, t) >= 0 \qquad \forall(\mathbf{v}, t) \in X. \tag{3-3-9}$$

In particular

$$\sigma(\nabla(\mathbf{u}_m + \mathbf{a}), \nabla\mathbf{v}) + (\mathbf{M}(\mathbf{u}_m + \mathbf{a}), \mathbf{v}) + (S_m + \psi_\delta, \mathbf{g}\cdot\mathbf{v}) = 0$$
$$\forall \mathbf{v} \in \mathbf{V} \times \{0\} \cap X,$$

$$(\mathrm{N}(\nabla(S_m + \psi_\delta), \nabla t) + R((\mathbf{u}_m + \mathbf{a})\cdot\nabla(S_m + \psi_\delta), t) = 0$$
$$\forall t \in \{0\} \times H_0^1(\Omega) \cap X.$$

Then (3-3-7) and (3-3-8) are satisfied and ξ_k^m and η_l^m could be determined through (3-3-6)-(3-3-8).

We multiply (3-3-7) by ξ_k^m, multiply (3-3-8) by η_l^m and add the corresponding equalities for $k, l = 1, \dots, m$. This gives

$$\begin{aligned} 0 &=< G_m(\mathbf{u}_m, S_m), (\mathbf{u}_m, S_m) > \\ &\geq (\gamma - \frac{R\delta}{2\pi\sqrt{n}}) \|(\mathbf{u}_m, S_m)\|^2_{\mathbf{V}\times H_0^1(\Omega)} \\ &- [n_l \|\psi_\delta\|_{H^1(\Omega)} + (\sigma + \frac{m_l}{\pi\sqrt{n}} + R\delta) \|\mathbf{a}\|_{\mathbf{H}^1(\Omega)}] \|(\mathbf{u}_m, S_m)\|_{\mathbf{V}\times H_0^1(\Omega)}. \end{aligned}$$

which yields

$$\|(\mathbf{u}_m, S)\|_{\mathbf{V}\times H_0^1(\Omega)} \leq r \tag{3-3-10}$$

with

$$r = \frac{n_l \|\psi_\delta\|_{H^1(\Omega)} + (\sigma + \frac{m_l}{\pi\sqrt{n}} + R\delta) \|\mathbf{a}\|_{\mathbf{H}^1(\Omega)}}{\gamma - \frac{R\delta}{2\pi\sqrt{n}}}. \tag{3-3-11}$$

Since the sequence $(\mathbf{u}_m, S_m)$ is uniformly bounded, which is independent of m, there exists a pair $(\mathbf{u}, S)$ in $\mathbf{V} \times H_0^1(\Omega)$ and a subsequence $m' \to \infty$ (we can still write m instead of m' for convenience), such that

$$(\mathbf{u}_m, S_m) \to (\mathbf{u}, S) \qquad \text{in the weak topology of } \mathbf{V} \times H_0^1(\Omega). \tag{3-3-12}$$

Moreover, the compactness imbedding theorem shows that

$$(\mathbf{u}_m, S_m) \to (\mathbf{u}, S) \qquad \text{in the strong topology of } \mathbf{L}^4(\Omega) \times L^4(\Omega). \tag{3-3-13}$$

Thus, passing to the limit in (3-3-9) with $m \to \infty$, we have

$$< G(\mathbf{u}, S), (\mathbf{v}, t) >= 0 \qquad \forall (\mathbf{v}, t) \in X. \tag{3-3-14}$$

A continuity argument finally shows that (3-3-14) holds for $(\mathbf{v}, t) \in \mathbf{V} \times H_0^1(\Omega)$ and $(\mathbf{u}, S)$ is a solution of (3-2-7).

4. The Regularity and Uniqueness.

In this section we shall discuss some regularity and uniqueness results of the solution. First we have the following result.

Theorem 4.1. *If* $(\mathbf{u}, S) \in \mathbf{V} \times H_0^1(\Omega)$ *is a solution of the problem (3-2-7), then* $\mathbf{u} \in \mathbf{H}^2(\Omega), S \in H^2(\Omega)$.

Proof. Since($\mathbf{u}$, S) is a solution of (3-2-7), it follows that (3-2-4) holds for any $\mathbf{v} \in \mathbf{V}$. The argument in section 2 shows that there exists a $p \in L^2(\Omega)$ such that

$$-\sigma\nabla^2(\mathbf{u}+\mathbf{a}) + \nabla p + \mathbf{M}(\mathbf{u}+\mathbf{a}) + (S+\psi_\delta)\mathbf{g} = 0$$

in the distribution sense.

Consider the problem

$$\sigma\nabla^2\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{3-4-1}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega. \tag{3-4-2}$$

We observe that $\mathbf{f} = \sigma\nabla^2\mathbf{a} - \mathbf{M}(\mathbf{u}+\mathbf{a}) - (S+\psi_\delta)\mathbf{g} \in \mathbf{L}^2(\Omega)$ since $\mathbf{a} \in \mathbf{H}^2(\Omega)$, $\zeta \in H^2(\Omega)$, and (3-4-1) is the same as a nonhomogeneous Stokes equation. Thus the regularity theory for the generalized Stokes problem (see Cattabriga 1961: p. 311) can be directly used here. This implies $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $p \in H^1(\Omega)$.

Also, (3-2-5) holds for any $t \in H_0^1(\Omega)$. We consider the Dirichlet problem

$$-div\,(\mathbf{N}\nabla S) = -R(\mathbf{u}+\mathbf{a}) \cdot \nabla(S+\psi_\delta) + div\,(\mathbf{N}\nabla\psi_\delta) \quad \text{in } \Omega, \tag{3-4-3}$$

$$S = 0 \qquad \text{on } \partial\Omega. \tag{3-4-4}$$

Since $\psi_\delta \in H^2(\Omega)$, $(\mathbf{u}+\mathbf{a}) \in \mathbf{H}^2(\Omega) \subset L^\infty(\Omega)$ and $[-R(\mathbf{u}+\mathbf{a}) \cdot \nabla(S+\psi_\delta) + div\,(\mathbf{N}\nabla\psi_\delta)] \in L^2(\Omega)$, the standard regularity theory of elliptic equations tells us $S \in H^2(\Omega)$ (see Gilbarg and Trudinger 1983).

We remark that the further regularity results of solutions $\mathbf{u}$ and S could be obtained by reiterating the same procedure as in the proof of theorem 4.1, provided additional conditions are imposed on boundary $\partial\Omega$, and on the boundary data $\mathbf{u}$ and ζ. We simply state the following result:

Theorem 4.2. *Let Ω be an open set in R^n with boundary of class C^∞ and let $a_i(x)$, $i = 1, \ldots, n$, $\zeta(x)$ be given in $C^\infty(\bar{\Omega})$. Then any solution $(\mathbf{u}, S)$ of problem (3-2-7) belongs to $\mathbf{C}^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega})$.*

Finally, we establish a uniqueness theorem.

Theorem 4.3. *There exists a unique solution of (3-2-7) if $R < (2 - \frac{1}{\pi\sqrt{n}})$.*

Proof. Let $(\mathbf{u}_1, S_1)$ and $(\mathbf{u}_2, S_2)$ be two solutions of (3-2-7). We have

$$\begin{aligned}
0 &= < G(\mathbf{u}_1, S_1) - G(\mathbf{u}_2, S_2), (\mathbf{u}_1 - \mathbf{u}_2, S_1 - S_2) > \\
&= \sigma \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}}^2 + \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{H}}^2 + \|S_1 - S_2\|_{H_0^1(\Omega)}^2 \\
&+ (\mathbf{u}_1 - \mathbf{u}_2, \mathbf{g}(S_1 - S_2)) + R(\mathbf{u}_1 - \mathbf{u}_2, (S_1 + \psi_\delta)\nabla(S_1 - S_2)) \\
&\geq \sigma \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}}^2 + (1 - \frac{1}{2\pi\sqrt{n}})\|(\mathbf{u}_1 - \mathbf{u}_2, S_1 - S_2)\|_{\mathbf{H}\times H_0^1(\Omega)}^2 \\
&- R\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{H}}\|S_1 - S_2\|_{H_0^1(\Omega)}\|S_1 + \psi_\delta\|_{L^\infty(\Omega)}.
\end{aligned} \tag{3-4-5}$$

From the regularity result of the solution it follows that $S_1 + \psi_\delta$ is a continuous function and a weak maximum principle produces $\|S_1 + \psi_\delta\|_{L^\infty(\Omega)} \leq 1$. Thus (3-4-5) could be rewritten as

$$\sigma\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}}^2 + (1 - \frac{1}{2\pi\sqrt{n}} - \frac{R}{2})\|S_1 - S_2\|_{H_0^1(\Omega)}^2 \leq 0$$

and hence $(\mathbf{u}_1, S_1) = (\mathbf{u}_2, S_2)$ provided $(1 - \frac{1}{2\pi\sqrt{n}} - \frac{R}{2}) > 0$.

The uniqueness theorem, as expected, thus imposes a restriction on the value of the Rayleigh number R.

5. Conclusions.

This chapter studied the existence, regularity and uniqueness of the weak solution for steady convection flow in a porous medium. The study uses techniques of Temam (1977) who has dealt with the Navier-Stokes equations. The Galerkin method is used to construct the approximate solution and to prove the the existence of solutions. The interesting results are given in Theorem 4.1 and Theorem 4.2 concerning the regularity of weak solution, and in Theorem 4.3 concerning the uniqueness of the solution. In particular, it is shown that a unique solution exists if $R < (2 - \frac{1}{\pi\sqrt{n}})$.

Chapter 4

Convective Instabilities in Anisotropic Porous Media

1. Introduction.

Hydrodynamic stability has been recognized as one of the central problems of fluid mechanics, since it has many applications in engineering, meteorology, oceanography, astrophysics and geophysics. The study of hydrodynamic stability, for example, is concerned with the minimum consumption of energy, when and how laminar flows break down, the construction of automation elements by fluid jets, and the aerodynamics of profiles in supersonic regime. As Landau and Lipfshitz (1959) have pointed out 'yet not every solution of the equations of the motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equation of fluid dynamics but also be stable'. Mathematically this means that a solution $\mathbf{u}$ of the Navier-Stokes equations can exist for any value of the Reynolds number R_e but it corresponds to the observed motion only for R_e smaller than some critical number R_{ec}. In this and the following chapter, we turn our attention to stability problems in porous media. First, we consider a problem of convective instability in anisotropic porous media within the framework of linear stability theory. In the subsequent chapter we employ a sophisticated energy method to study the stability of a rotating Bénard problem in porous media.

There exists extensive literature on flow and heat transfer problems through porous media due to their numerous applications in different fields. While most analytical studies deal primarily with the mathematical formulation based upon Darcy's law and isotropic permeability, considerable interest, in the recent past, has been given to either adopting a non-Darcian model with isotropic porous medium, or using Darcy's law with anisotropic permeability. It is believed that porous medium with high permeability and rigid boundaries the Brinkman (1947) equation gives preferable results. Katto and Masuoka (1967) as well as Walker and Homsy (1977) used the Brinkman model for the studies of free convection of liquids in a porous medium bounded between parallel plates and heated from below. Vafai and Tien (1981) have extended the Brinkman model to include the inertial effect and solved, numerically, the problem of forced convection along a heated plate embedded in a porous medium. Brinkman's model has also been employed by Rudraiah and Masuoka (1982), in studying convective heat transfer in a horizontal porous layer heated from below with a free surface at the top, by Hsu and Cheng (1985), in investigating the boundary effects in a natural convection porous layer adjacent to a semi-infinite vertical plate with a power law variation of wall temperature, and by Vafai and Kim (1989), in considering a fully developed forced convection in a porous channel bounded by parallel plates.

On the other hand, papers on thermal convection in an anisotropic porous medium are not numerous as compared to the number of publications on isotropic porous media. Castinel and Combarnous (1974) conducted an experimental and theoretical investigation on the stability criterion for porous media with anisotropic permeability. Epherre (1975) extended the analysis to account for thermal anisotropy. Kvernvold and Tyvand (1979) analyzed the supercritical roll motion, its stability and associated heat transfer. In more recently published work, McKibbin (1986) studied the effects of anisotropy on the convective stability of a porous layer, considering boundary conditions sufficiently general enough to allow flow through the top. Nilsen and Storesletten (1990) gave an analytical study on natural convection in horizontal rectangular channels filled with isotropic and anisotropic porous media. Tyvand and Storesletten (1991) investigated the onset of convection in an anisotropic porous medium with oblique principal axes. Chen et al. (1991) considered the onset of thermal convection in a system consisting of a fluid layer overlying a porous layer with anisotropic permeability and thermal diffusivity.

In the present study, we consider the Rayleigh-Bénard instability in a porous layer based upon both the Brinkman model and anisotropic permeability. The general Brinkman equation with anisotropic permeability could be used to describe sparsely packed fibrous insulation materials, where convection currents may occur. The marginal stabilities are analyzed for both free and rigid boundary conditions. We give general analysis in the case of free-free surfaces, but unlike the case of isotropic permeability we have to limit our discussion to the two-dimensional equation in the case of rigid-rigid surfaces. Fortunately, this is sufficient to determine the critical Rayleigh numbers for the onset of convection. At rigid boundaries, conducting and insulating cases are considered separately. We remark that, from our analysis, which is likely to be useful for moderate porosity materials, we can derive limiting results for low porosity Darcy approximation as well as for ordinary viscous fluid.

2. Mathematical Formulation.

We consider a fluid-saturated porous layer which is bounded above and below by two infinite and impermeable horizontal planes. The upper and lower boundaries are separated by a distance h and are at constant temperature T_0 and $T_0 + \Delta T$, respectively. The layer is heated from below so that the characteristic temperature difference ΔT is positive. For simplicity, the saturated porous medium is assumed to have coinciding principal axes of permeability; the thermal conductivity, however, is assumed to be isotropic. One of these axes is directed upwards, in the z direction. The x and y axes are defined by the directions of the other two principal axes. k_1, k_2 and k_3 are the components of the permeability in the x, y and z directions,

$$\sigma\Delta^2 w + (\xi_1 \frac{\partial^2 u}{\partial x \partial z} + \xi_2 \frac{\partial^2 v}{\partial y \partial z} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}) = -R(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}). \qquad (4\text{-}2\text{-}5)$$

If the porous material is horizontally isotropic, i.e., $\xi_1 = \xi_2 = \xi$, equations (4-2-4) as well as (4-2-5) reduce to the same equation

$$-\sigma\Delta^2 w + (\Delta_1 w + \xi \frac{\partial^2 w}{\partial z^2}) = R\Delta_1 \theta\,, \qquad (4\text{-}6)$$

where $\Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In the isotropic case, $\xi_1 = \xi_2 = 1$, and (4-2-6) reduces to the well known equation

$$-\sigma\Delta^2 w + \Delta w = R\Delta_1 \theta\,. \qquad (4\text{-}2\text{-}7)$$

For anisotropic porous medium, by standard manipulations we eliminate the horizontal velocities from (4-2-4) and (4-2-5). The resulting equation can be written as

$$\begin{aligned}
&\sigma\Delta[(\xi_2+1)\frac{\partial^2 w}{\partial x^2} + (\xi_1+1)\frac{\partial^2 w}{\partial y^2} + (\xi_1+\xi_2)\frac{\partial^2 w}{\partial z^2}] \\
&\quad - [\xi_2 \frac{\partial^2 w}{\partial x^2} + \xi_1 \frac{\partial^2 w}{\partial y^2} + \xi_1\xi_2 \frac{\partial^2 w}{\partial z^2}] - \sigma^2\Delta^3 w \\
&= R[(\sigma\Delta - \xi_2)\frac{\partial^2 \theta}{\partial x^2} + (\sigma\Delta - \xi_1)\frac{\partial^2 \theta}{\partial y^2}]. \qquad (4\text{-}2\text{-}8)
\end{aligned}$$

As we remarked earlier the linearized version of equations (4-2-1) to (4-2-3) are self-adjoint, and onset is thus by the neutral mode $\partial/\partial t = 0$, which reduces (4-2-3) to

$$\Delta\theta = -Rw\,. \qquad (4\text{-}2\text{-}9)$$

An eigenvalue problem could be formed from (4-2-8) and (4-2-9) with appropriate boundary conditions. It is of some interest here to point out a difference from the isotropic case. We note that in the elimination of horizontal velocities the order of differential equation (4-2-8) has increased and thus, in principle, an extra boundary condition is needed for the vertical velocity to make the eigenvalue problem solvable. However, by applying some physical arguments, the problem can be made tractable. We shall discuss some of these situations in the next section.

3. Marginal Stability Analysis.

(i) Stress Free Boundary Surface

If the boundary conditions associated with the governing equations (4-2-1) to (4-2-3) are given by free surface conditions, we must have

$$\theta = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \qquad \text{at} \quad z = 0, 1\,. \tag{4-3-1}$$

Introducing normal modes for w and θ as

$$w = W(z)e^{i(lx+my)}\,,$$

$$\theta = \theta(z)e^{i(lx+my)}\,,$$

and using continuity equations (4-2-2) and (4-2-4) or (4-2-5) yield

$$\frac{d^2W}{dz^2} = \frac{d^4W}{dz^4} = 0 \qquad \text{at} \quad z = 0, 1\,. \tag{4-3-2}$$

This suggests that $W(z) = \sin n\pi z$. The Rayleigh number for the onset of convection is easily found to be

$$\begin{aligned} R^2 &= \sigma\,(l^2+m^2+n^2\pi^2)^2 + (l^2+m^2+n^2\pi^2) \\ &+ n^2\pi^2\frac{(l^2+m^2+n^2\pi^2)[\sigma(l^2+m^2+n^2\pi^2)+\xi_1][\sigma(l^2+m^2+n^2\pi^2)+\xi_2]}{\sigma(l^2+m^2+n^2\pi^2)(l^2+m^2)+(\xi_2 l^2+\xi_1 m^2)}. \end{aligned} \tag{4-3-3}$$

We can check expression (4-3-3) in both limits of $\sigma \to 0$, the Darcy approximation with anisotropy, and $\sigma \to \infty$, the Bénard problem for an ordinary viscous fluid. In the case $\sigma = 0$, equation (4-3-3) is reduced to

$$R^2 = \frac{(l^2+m^2+n^2\pi^2)\xi_1\xi_2 n^2\pi^2}{\xi_2 l^2 + \xi_1 m^2} + (l^2+m^2+n^2\pi^2)\,, \tag{4-3-4}$$

and the corresponding critical Rayleigh number is

$$R_c^2 = \pi^2(\min(\xi_1^{1/2}, \xi_2^{1/2}) + 1)^2, \qquad (4\text{-}3\text{-}5)$$

which coincides with the results given by Kvernvold and Tyvand (1979). In the case $\sigma \to \infty$ ($\phi = 1$ as $\sigma \to \infty$), (4-3-3) reduces to the classical results (cf. Chandrasekhar 1961)

$$\bar{R}^2 = \frac{(a^2 + n^2\pi^2)^3}{a^2} \qquad \bar{R}_c^2 = \frac{27}{4}\pi^4 = 657.5, \qquad (4\text{-}3\text{-}6)$$

where $a^2 = l^2 + m^2$, $\bar{R} = (\frac{\alpha\beta g h^4}{\kappa\nu})^{1/2}$. For small gaps or moderately permeable materials σ may lie between these two extremes. In order to get the critical Rayleigh number R_c^2, we must minimize equation (4-3-3) with respect to l^2 and m^2, and $R_c^2 = min\, R^2$. Being aware that

$$\frac{\partial R^2}{\partial l^2} - \frac{\partial R^2}{\partial m^2} = n^2\pi^2(\xi_1 - \xi_2)(l^2 + m^2 + n^2\pi^2)\times$$
$$\frac{[\sigma(l^2 + m^2 + n^2\pi^2) + \xi_1][\sigma(l^2 + m^2 + n^2\pi^2) + \xi_2]}{\{[\sigma(l^2 + m^2 + n^2\pi^2) + \xi_2]l^2 + [\sigma(l^2 + m^2 + n^2\pi^2) + \xi_1]m^2\}^2}, \qquad (4\text{-}3\text{-}7)$$

and taking the lower bound value for R^2, we can observe that $\frac{\partial^2 R}{\partial l^2}$ and $\frac{\partial^2 R}{\partial m^2}$ cannot be zero simultaneously at any point of the open domain $\{(l^2, m^2) \mid 0 < L^2 < \infty,\quad 0 < m^2 < \infty\}$ except when $\xi_1 = \xi_2$. Thus R^2 can only reach its minimum either at $m^2 = 0$ or $l^2 = 0$. Accordingly the critical wave number is obtained by one of two conditions:

$$m^2 = 0, \quad \frac{dR^2}{dl^2} = 0 \qquad \text{or} \qquad l^2 = 0, \quad \frac{dR^2}{dm^2} = 0, \qquad (4\text{-}3\text{-}8)$$

depending upon which of ξ_1 and ξ_2 is smaller. The critical Rayleigh number for the onset of convection then becomes

$$R_c^2 = \sigma(\bar{x} + \pi^2)^2 + (\bar{x} + \pi^2) + \frac{\pi^2}{\bar{x}}(\bar{x} + \pi^2)[\sigma(\bar{x} + \pi^2) + \min(\xi_1, \xi_2)], \qquad (4\text{-}3\text{-}9)$$

where $\bar{x}^{\frac{1}{2}}$ is a wave number and $\bar{x}$ is the smallest root of equation

$$2\sigma x^3 + x^2(3\sigma\pi^2 + 1) - \pi^4(\sigma\pi^2 + \min(\xi_1, \xi_2)) = 0 . \qquad (4\text{-}3\text{-}10)$$

Letting $\sigma \to 0$ in (4-3-9) and (4-3-10) yields $\bar{x} = \pi^2 [\min(\xi_1, \xi_2)]^{\frac{1}{2}}$, $R_c^2 = \pi^2 [1 + \min(\xi_1^{\frac{1}{2}}, \xi_2^{\frac{1}{2}})]^2$. Also the asymptotic values in (4-3-9) and (4-3-10), as $\sigma \to \infty$, are

$$\bar{x} = \frac{1}{2}\pi^2 , \qquad \bar{R}_c^2 = \frac{27}{4}\pi^4 .$$

The critical numbers in equations (4-3-5) and (4-3-6) are thus recovered for both limiting cases. Unlike the isotropic and horizontal isotropic cases ($\xi_1 = \xi_2$), there is a specific direction along which rolls are preferred because of losing the horizontal symmetry. Rolls are aligned in the y-direction if $\xi_1 < \xi_2$, or in the x-direction otherwise.

Figure 1 shows numerical calculations for Eqs. (4-3-9) and (4-3-10). In it we have plotted the effective critical Rayleigh number $\bar{R}_c^2$ against the effective Darcy's parameter h^2/k_3 for two different ratios of vertical permeability, ξ_1 (or ξ_2). We note that $\bar{R}_c^2$ takes values close to $\pi^2(1 + \xi^{\frac{1}{2}})^2 h^2/k_3$ for $h^2/k_3 > 10^3$ and approaches the number 657 for $h^2/k_3 < 10^{-1}$. Thus there exists a range $10^{-1} \leq h^2/k_3 \leq 10^3$ for which the stability criteria is intermediate to the two limiting cases. We also observe that the critical Rayleigh number increases as ξ_1 (or ξ_2) increases.

(ii) <u>Rigid Boundary Surface</u>

On the rigid boundary, the usual no-slip boundary conditions will be imposed for the velocity, so that $w = Dw = 0$ at $z = 0, 1$, where $D = \frac{d}{dz}$. For the perturbation of temperature, both insulating boundary ($D\theta = 0$) and conducting boundary ($\theta = 0$) are considered, respectively. For solving a desired eigenvalue problem we cannot use (4-2-8) in its present form which is based on full three dimensional equations, because one extra boundary condition for the vertical velocity is not available. However, to determine the critical Rayleigh number and patterns of convection, we fortunately only need to consider a two-dimensional case instead of a three-dimensional case in an anisotropic porous medium. In this regard, we follow the physical argument pointed out by Tyvand and Storesletten (1991): 'At the onset of convection the preferred flow cells tend to arrange themselves such that the tangential permeability along the streamlines is as large as possible'. We have already seen that in the case of free-free boundaries such an argument agrees with the mathematical conclusion. By starting with the full three-dimensional equations we were able to show that the critical Rayleigh number could only be reached at its minimum either when $m^2 = 0$ or $l^2 = 0$. This physically implies that the onset of convection takes place in the case when there is no fluid motion either in the x-direction or in the y-direction. Thus, when $\xi_1 < \xi_2$ it is clear that motion in the y-direction should be suppressed, as its permeability is minimal. In such a case we expect convection cells in the (x, z) plane, independent of y. When $\xi_1 > \xi_2$, we have maximum permeability in the y -direction. This indicates that the preferred motion is independent of x. Now suppose $\xi_1 < \xi_2$, so that the motion is assumed independent of y. Equation (4-2-4) and (4-2-5) can be written as

$$\sigma(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})^2 w + (-\xi_1 \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2})w = -R\frac{\partial^2 \theta}{\partial x^2}. \qquad (4\text{-}3\text{-}11)$$

When normal modes $\theta = \theta(z)e^{ilx}, \quad w = w(z)e^{ilx}$ are introduced, we have

$$[\sigma(D^2 - l^2)^2 - (\xi_1 D^2 - l^2)]w = Rl^2\theta, \qquad (4\text{-}3\text{-}12)$$

together with

$$(D^2 - l^2)\theta = -Rw, \qquad (4\text{-}3\text{-}13)$$

where $D = \frac{d}{dz}$. When $\xi_1 > \xi_2$ we get the same form as (4-3-12) and (4-3-13) by replacing ξ_1, l^2 with ξ_2, m^2, respectively. We shall discuss two cases:

(a) Rigid and Insulating Boundaries

The requirements of this boundary condition yield

$$w = Dw = D\theta = 0\,, \quad \text{at} \quad z = 0, 1\,.$$

Nield (1975) was the first author to study the case of constant-flux boundary conditions. Physically these are interesting in their own right as a limiting case of some approximated practical situations. Mathematically, they allow a solution of an eigenvalue problem in simple closed form. This means that the Galerkin method which usually yields only an approximate value of R^2, now yields an exact value when the wave number equals to zero. We apply the Galerkin method by using the single terms $w = Aw_1$, $\theta = B\theta_1$, where w_1 and θ_1 are suitably chosen as the trial functions that satisfy the given boundary conditions. Substituting them into (4-3-12) and (4-3-13), multiplying the first equation by w_1, the second equation by θ_1, integrating by parts, eliminating A, B and dropping the suffixes, we have

$$R^2 = \frac{<\sigma(D^2w)^2 + (\xi_1 + 2\sigma l^2)(Dw)^2 + (\sigma l^4 + l^2)w^2><(D\theta)^2 + l^2\theta^2>}{l^2 <\theta w>^2}, \tag{4-3-14}$$

where $<f>$ denotes $\int_0^1 f\,dz$.

Following the standard perturbation procedure for sufficiently small wave number, we can choose the appropriate trial functions as

$$\theta = 1\,, \tag{4-3-15}$$

$$w = A\cosh P_0 z + B\sinh P_0 z + Cz + E - \frac{1}{2\xi_1}z^2\,, \tag{4-3-16}$$

where w is the exact solution of boundary value problem

$$(\sigma D^4 - \xi_1 D^2)w = 1, \tag{4-3-17}$$

with

$$w = Dw = 0 \quad \text{at} \quad z = 0, 1$$

and

$$\begin{aligned} A = -E &= \frac{P_0 + P_0 \cosh P_0 - 2 \sinh P_0}{2\xi_1 P_0 [2 - 2\cosh P_0 + P_0 \sinh P_0]}, \\ B = -\frac{C}{P_0} &= -\frac{1}{2\xi_1 P_0}, \qquad P_0 = \xi_1/\sigma . \end{aligned} \tag{4-3-18}$$

Using equations (4-3-14) to (4-3-18), we obtain the critical Rayleigh number

$$R_c^2 = \frac{1}{< w >} = \frac{12\xi_1 P_0^2 (2 - 2\cosh P_0 + P_0 \sinh P_0)}{(24 - 4P_0^2) - (24 + 8P_0^2)\cosh P_0 + (24P_0 + P_0^3)\sinh P_0}. \tag{4-3-19}$$

This gives the asymptotic value

$$R_c^2 \sim \xi_1 (12 + 72 P_0^{-1}) \qquad \text{as} \quad P_0 \to \infty. \tag{4-3-20}$$

In the case $\xi_1 > \xi_2$, when the motion is independent of x, we can obtain the analogous results as in (4-3-19) and (4-3-20) by replacing ξ_1 with ξ_2. Thus, combining both cases, we may write

$$R_c^2 = \frac{12 P_0^2 (2 - 2\cosh P_0 + P_0 \sinh P_0) \min(\xi_1, \xi_2)}{(24 - 4P_0^2) - (24 + 8P_0^2)\cosh P_0 + (24P_0 + P_0^3)\sinh P_0}, \tag{4-3-21}$$

$$R_c^2 \sim \min(\xi_1, \xi_2)(12 + 72 P_0^{-1}) \quad \text{as} \quad P_0 \to \infty . \tag{4-3-22}$$

We have noted that Rudraiah *et al.* (1980) studied the Brinkman equation associated with rigid and insulating boundaries in isotropic porous medium, but they did not use the proper trial functions causing a large discrepancy. This led Nield (1983) to question the applicability of Brinkman's equation in the bulk of the porous medium. Later Nield (1985) gave a more reasonable explanation that the discrepancy was not due to the use of Brinkman's equation, but rather due to the manner in which the Galerkin approximation was used. When $\xi_1 = \xi_2 = 1$, the results expressed in (4-3-21) and (4-3-22) coincide with the results obtained by Nield (1985).

From equations (4-3-16) and (4-3-18), we note that

$$w(z) \sim \frac{1}{2}\frac{1}{\min(\xi_1,\xi_2)}\{z - z^2 + P_0^{-1}[e^{P_0(z-1)} + e^{-P_0 z} - 1]\}\,. \qquad (4\text{-}3\text{-}23)$$

The boundary layer effect can be demonstrated from above equation.

(b) Rigid and Conducting Boundaries

In this case the boundary conditions are

$$w = Dw = \theta = 0 \qquad \text{at} \quad z = 0, 1.$$

With these conditions, we solve the equations

$$[\sigma(D^2 - a^2)^2 - \min(\xi_1,\xi_2)D^2 + a^2]\,w \;=\; Ra^2\theta\,, \qquad (4\text{-}3\text{-}24)$$

$$(D^2 - a^2)\,\theta \;=\; -Rw\,, \qquad (4\text{-}3\text{-}25)$$

numerically by using the Rayleigh-Ritz method.

We expand θ in a sine series in the form

$$\theta = \sum_m A_m \sin m\pi z\,, \qquad (4\text{-}3\text{-}26)$$

and assume for w an expansion of the form

$$w = \frac{1}{\sigma}Ra^2 \sum_m A_m W_m(z)\,. \qquad (4\text{-}3\text{-}27)$$

Without loss of generality we assume $\xi_1 < \xi_2$, thus W_m are required to satisfy the equation

$$[\,(D^2 - a^2)^2 - P_0 D^2 + \frac{P_0}{\xi_1}a^2\,]\,W_m = \sin m\pi z\,, \qquad (4\text{-}3\text{-}28)$$

with

$$W_m = DW_m = 0\,, \qquad \text{at} \quad z = 0, 1\,, \qquad (4\text{-}3\text{-}29)$$

where

$$P_0 = \frac{\xi_1}{\sigma} = \frac{h^2}{k_1}\phi .$$

The general solution of (4-3-28) is

$$\begin{aligned} W_m = \; & P_m \cosh r_1 z + Q_m \sinh r_1 z \\ & + L_m \cosh r_2 z + R_m \sinh r_2 z + \gamma_m \sin m\pi z , \end{aligned} \tag{4-3-30}$$

where r_1^2 and r_2^2 are roots of equation

$$r^4 - (2a^2 + P_0)r^2 + (\frac{P_0}{\xi_1} + a^2)a^2 = 0 , \tag{4-3-31}$$

and

$$\gamma_m = \frac{1}{(m^2\pi^2 + a^2)^2 + \frac{P_0}{\xi_1}(\xi_1 m^2\pi^2 + a^2)} . \tag{4-3-32}$$

The requirement of boundary conditions (4-3-29) implies

$$P_m = -L_m = \gamma_m m\pi f_1(r_1, r_2)/f_4(r_1, r_2) , \tag{4-3-33}$$

$$Q_m = \gamma_m m\pi f_2(r_1, r_2)/f_4(r_1, r_2) , \tag{4-3-34}$$

$$R_m = \gamma_m m\pi f_3(r_1, r_2)/f_4(r_1, r_2) , \tag{4-3-35}$$

where

$$\begin{aligned} f_1(r_1, r_2) = & r_1 \cosh r_1 \sinh r_2 - r_2 \sinh r_1 \cosh r_2 \\ & + (-1)^m (r_2 \sinh r_1 - r_1 \sinh r_2) , \end{aligned} \tag{4-3-36}$$

$$\begin{aligned} f_2(r_1, r_2) = & r_2 \cosh r_1 \cosh r_2 - r_1 \sinh r_1 \sinh r_2 - r_2 \\ & + (-1)^m (r_2 \cosh r_2 - r_2 \cosh r_1) , \end{aligned} \tag{4-3-37}$$

$$\begin{aligned} f_3(r_1, r_2) = & r_1 \cosh r_1 \cosh r_2 - r_2 \sinh r_1 \sinh r_2 - r_1 \\ & + (-1)^m (r_1 \cosh r_2 - r_1 \cosh r_1) , \end{aligned} \tag{4-3-38}$$

$$\begin{aligned} f_4(r_1, r_2) = & 2r_1 r_2 + (r_1^2 + r_2^2) \sinh r_1 \sinh r_2 \\ & - 2r_1 r_2 \cosh r_1 \cosh r_2 . \end{aligned} \tag{4-3-39}$$

On substituting (4-3-26) and (4-3-27) into (4-3-25), multiplying by $\sin n\pi z$ on both sides and integrating by parts, we obtain

$$\sum_m \left[\frac{1}{2}(m^2\pi^2 + a^2)\delta_{mn} - \frac{1}{\sigma}R^2a^2(n \mid m)\right]A_m = 0\,, \quad (n = 1,2,\dots)\,, \qquad \text{(4-3-40)}$$

where matrix $(n \mid m)$ has the form

$$\begin{aligned}(n \mid m) &= \int_0^1 W_m(z)\sin n\pi z\,dz \\ &= P_m \frac{n\pi}{n^2\pi^2 + r_1^2}[1 + (-1)^{n-1}\cosh r_1] \\ &+ Q_m \frac{n\pi}{n^2\pi^2 + r_1^2}[(-1)^{n-1}\sinh r_1] \\ &+ L_m \frac{n\pi}{n^2\pi^2 + r_1^2}[1 + (-1)^{n-1}\cosh r_2] \\ &+ R_m \frac{n\pi}{n^2\pi^2 + r_1^2}[(-1)^{n-1}\sinh r_2] + \frac{1}{2}\gamma_m\delta_{mn} \\ &= \gamma_m\gamma_n mn\pi^2(r_2^2 - r_1^2)\frac{f_5(r_1,r_2,m,n)}{f_4(r_1,r_2)} + \frac{1}{2}\gamma_m\delta_{mn}\,,\end{aligned} \qquad \text{(4-3-41)}$$

with

$$\begin{aligned}f_5(r_1,r_2,m,n) = {}&(r_1\cosh r_1\sinh r_2 - r_2\cosh r_2\sinh r_1)(1 + (-1)^{m+n-2}) \\ &(r_1\sinh r_2 - r_2\sinh r_1)((-1)^{n-1} + (-1)^{m-1}).\end{aligned} \qquad \text{(4-3-42)}$$

Equation (4-3-40) forms an algebraic eigenvalue problem, which is solved numerically. It is interesting to look at the first approximation. This gives an explicit expression

$$\bar{R}^2 = (\pi^2 + a^2)/\{2a^2\phi[\frac{1}{2}\gamma_1 + \gamma_1^2\pi^2(r_2^2 - r_1^2)f_5(r_1,r_2,1,1)/f_4(r_1,r_2)]\}\,, \qquad \text{(4-3-43)}$$

where

$$\bar{R}^2 = \frac{\alpha\beta g h^4}{\kappa\nu}\,.$$

The two asymptotic limits can be derived from (4-3-43). When P_0 is very small ($P_0 \to 0$ and in this case $\phi = 1$), one has

$$R^2 = \frac{(\pi^2 + a^2)^3}{a^2\{1 - 16a\pi^2 \cosh \frac{1}{2}a/[(\pi^2 + \pi^2)^2(\sinh a + a)]\}}, \tag{4-3-44}$$

which recovers the corresponding result for viscous flow obtained by Chandrasekhar (1961). When P_0 is very large ($P_0 \to \infty$), an asymptotic limit gives

$$\bar{R}^2 = \frac{h^2}{k_3}\frac{(\pi^2 + a^2)(\xi_1\pi^2 + a^2)}{a^2}. \tag{4-3-45}$$

The critical number is thus given by

$$\bar{R}_c^2 = \frac{h^2}{k_3}\pi^2(1 + \xi_1^{\frac{1}{2}})^2. \tag{4-3-46}$$

The main numerical results are shown in Fig. 2. It is seen that for low permeabilities, (large h^2/k_3), the Darcy limit $\bar{R}_c^2 = \pi^2(1 + \xi_1^{\frac{1}{2}})^2$ is reached for $h^2/k_3 > 10^3$. Similarly for high permeabilities $h^2/k_3 < 10^{-1}$, the response is close to that of a viscous fluid $\bar{R}_c^2 \sim 1708$. Thus there exists a range $10^{-1} < h^2/k_3 < 10^3$ for which the criteria for instability is intermediate to the two asymptotic limits. In this respect our conclusion is analogous to that drawn by Walker and Homsy (1977) in the isotropic case, but it is obvious that the criterion in the present case is also affected by ξ_1 (or ξ_2) which is the ratio of vertical permeability with horizontal permeability.

4. Conclusions.

In this chapter we presented a theoretical analysis of thermal convection in a porous layer using Brinkman's equation with anisotropic permeability. For both free and rigid boundaries the criteria for the onset of convection has been derived. In the case of stress free boundaries there is a specific direction along which rolls are preferred. The alignment is dependent upon the magnitude of ratios of the permeabilities ξ_1 and ξ_2. As to the stability criterion it is found that there exists a particular range, as shown in Fig. 1, for which the criterion is intermediate between the two limiting cases.

For rigid boundaries the cases of insulating boundary and conducting boundary are discussed separately. In the case of insulating boundary a single term Galerkin method appears to give reasonable results. Here the values of critical Rayleigh number depend upon ξ_1 and ξ_2 and in limiting cases of low and high permeabilities give compatible results. In the case of a conducting boundary, the Rayleigh-Ritz method is employed and the resulting algebraic eigenvalue problem is solved numerically. Here again the critical numbers depend upon ξ_1, ξ_2 and the Darcy number h^2/k_3. From Fig. 2, we note that there exists a range $10^{-1} < h^2/k_3 < 10^3$ for which the instability criteria is intermediate between the asymptotic limits of Darcy's law and ordinary viscous fluid.

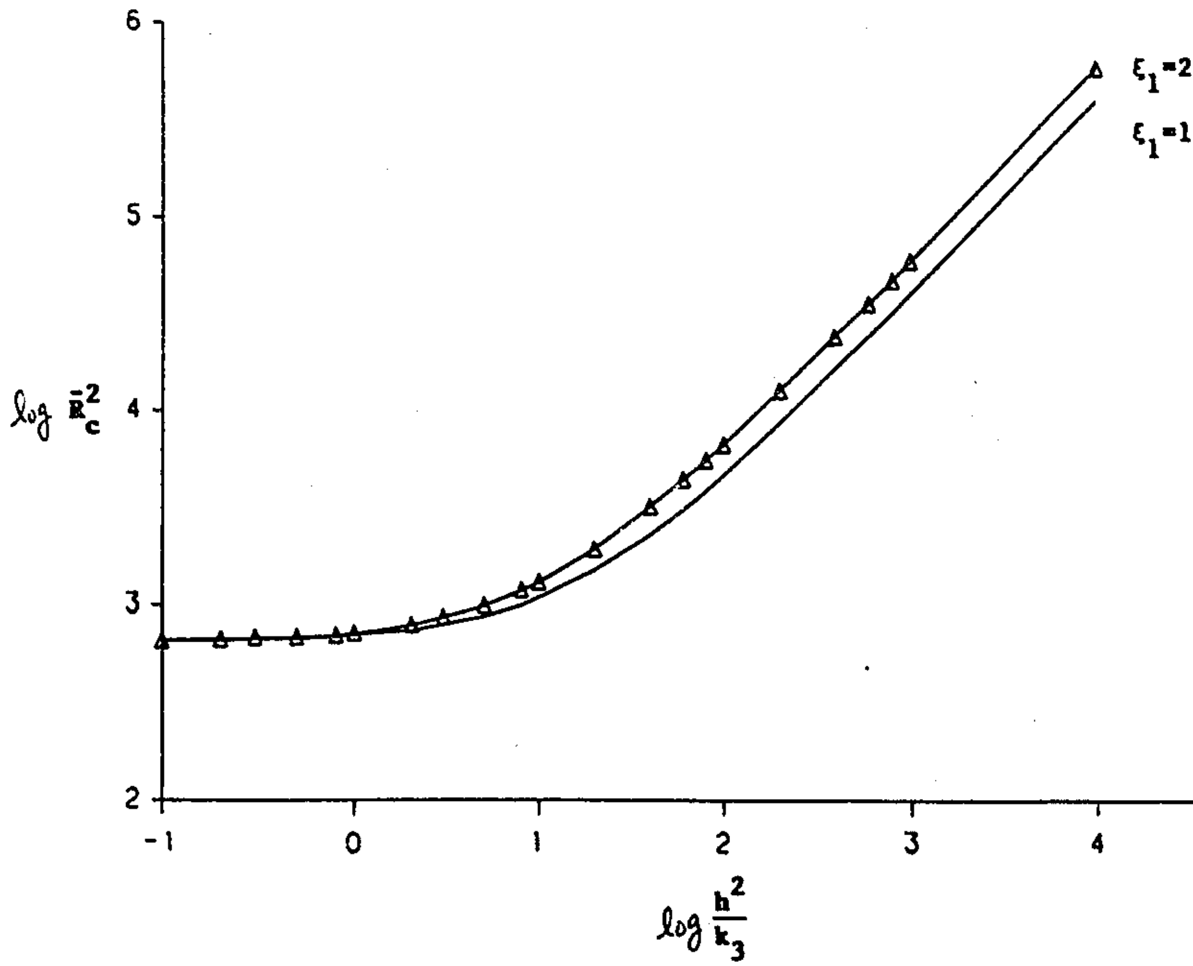

Fig.1. The critical Rayleigh number as a function of dimensionless permeability $\frac{h^2}{k_3}$ and ratio of vertical permeability with horizontal permeability ξ_1, $\xi_1 = 1$ is the isotropic case.

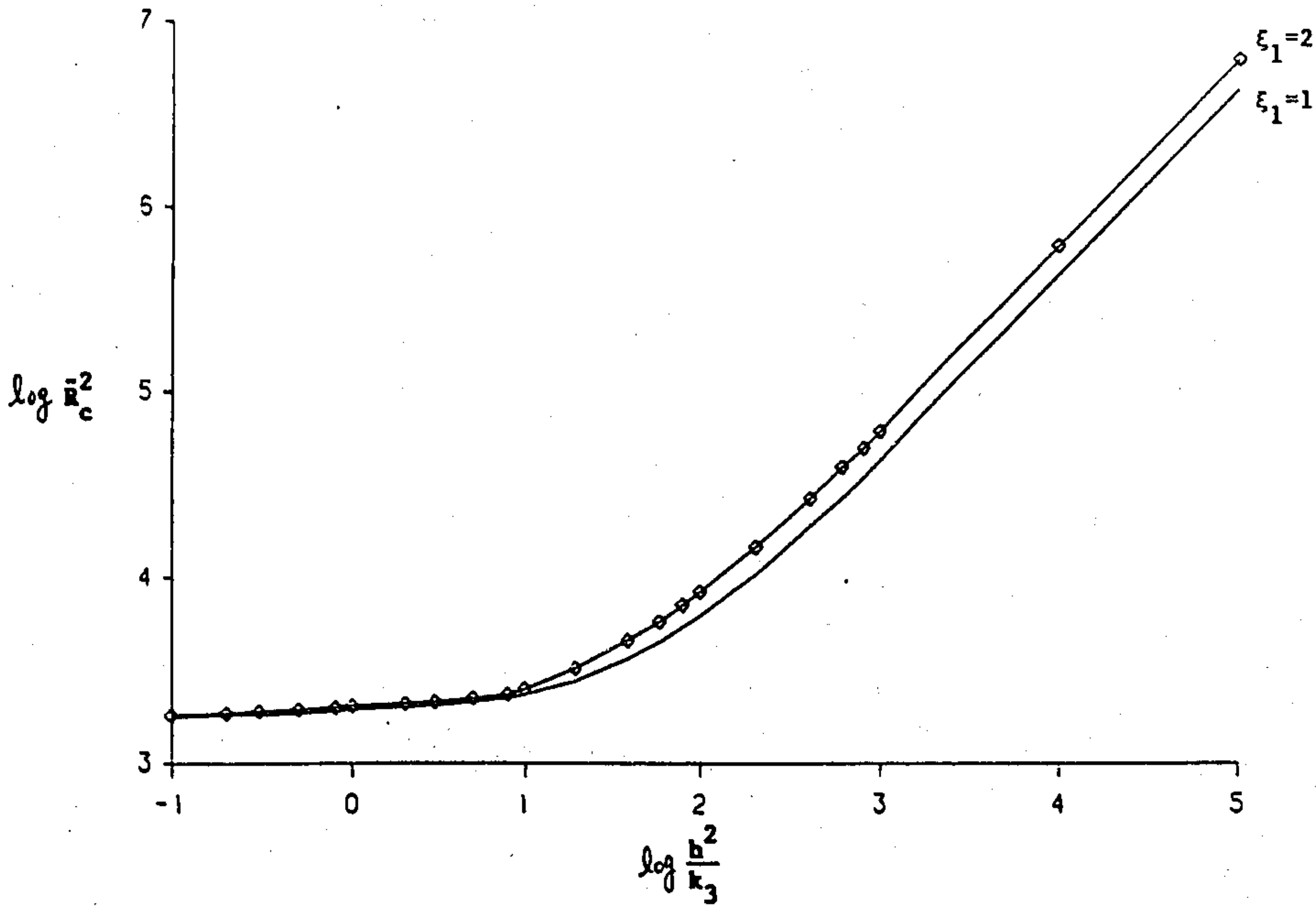

Fig.2. **The critical Rayleigh number as a function of $\frac{h^2}{k_3}$ and ξ_1 in the case of rigid and conducting boundary conditions.**

Chapter 5

Energy Stability for the Rotating Porous Bénard Problem

1. Introduction.

In hydrodynamic stability theory there are two basic methods, namely, the linear stability method and the energy method, which have been used extensively. The two methods complement each other. While the linear stability method determines a critical dimensionless bound above which the disturbances of a basic flow are unstable, the energy method predicts a critical bound below which a basic flow is asymptotically stable (in the Liapunov sense). However, in some cases when the linearized system of governing equations turns out to be symmetric, the linear stability and energy stability predictions coincide (cf. Davis, 1969, Galdi and Straughan, 1985).

The energy method is essentially due to Orr (1907), but its recent revival has been inspired by the creative work of Serrin (1959) and Joseph (1965, 1966, 1976). Despite the unquestionable success of the energy method in several stability problems, there is some definite skepticism about its ongoing use. One of the weakening features comes from a typical example of the stability of the Bénard problem between rotating parallel, horizontal planes. The classical energy theory of Serrin and Joseph fails to predict correctly, by several orders of magnitude, the effects of inhibiting convection because of rotation, which are clearly revealed by both linear theory and experiments. To overcome such discrepancies, rapid improvements of energy theory have been made in recent years. Galdi and Straughan (1985) and Mulone and Rionero (1989) have successfuly improved the classical energy theory by proposing generalized energy functionals. In both papers, these authors again studied the above problem and drew conclusions which were in close agreement with the results found by linear theory and experiments. This new approach to energy theory is summarized and further analysed by Galdi and Padula (1990). In this remarkable paper, the authors have singled out systems for which the new theory furnishes necessary and sufficient conditions for stability, and have provided an algorithm to construct the appropriate generalized energy functionals for a wide class of physical problems. With regard to the rotating Bénard problem, we note that an energy function, using only L^2 norm of the disturbance variables may not be good enough to obtain desired results. The generalized energy functionals, with stronger norm, usually $W^{1,2}$ or equivalent to that, would be useful in the study of the nonlinear stability of the rotating Bénard problem.

The stability problems of convective flow in porous media using the energy method have been investigated by Westbrook (1969), Joseph (1976), Homsy and Sherwood (1976), and Rionero and Straughan (1990). To our knowledge, nonlinear stability of the rotating Bénard problem in porous media has not been examined so far. In this chapter we employ Brinkman's model as a suitable prototype for high porosity porous media and based on it we are able to apply the theory of Galdi and Padula (1990) in studying the rotating Bénard porous problem.

In this chapter, after presenting the basic equations, the evolution equation of an energy functional is derived. Adopting the same energy functional as constructed by Galdi and Padula for the viscous rotating Bénard problem, the a priori estimation of the energy functional is then given which will ensure the nonlinear stability under some sufficient conditions. This leads to the variational problem for the critical Rayleigh number of energy theory. For comparison purposes, the linear instability is analyzed briefly. The chapter ends with numerical results and the discussion.

2. Perturbation Equations.

Let us consider an infinite horizontal porous layer saturated with a homogeneous fluid under the action of a vertical gravity field $\mathbf{g} = -g\mathbf{k}$ in which an adverse temperature gradient $\beta > 0$ is maintained. We also assume that the fluid is rotating about the vertical axis z with a constant angular velocity Ω. A perturbation of velocity $\mathbf{u}$, temperature θ and pressure field p is taken about the nonconvecting stationary solution $\mathbf{u} = 0$, $T = -\beta z + T_0$. The fluid is assumed to be contained in a porous medium between two planes $z = 0$ and $z = h$ with assigned temperatures $T = T_0$ at $z = 0$; $T = T_1$ at $z = h$, where $T_0 > T_1$ and the temperature gradient across the fluid layer is $\beta = (T_0 - T_1)/h$.

The momentum, continuity and energy equations for the porous fluid layer in the Boussinesq approximation are, respectively (Walker and Homsy, 1977)

$$\mathbf{u}_t + \frac{1}{\phi}\mathbf{u} \cdot \nabla \mathbf{u} = \Delta \mathbf{u} - P_0 \mathbf{u} + R\theta \mathbf{k} + T\mathbf{u} \times \mathbf{k} - \nabla p, \tag{5-2-1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{5-2-2}$$

$$P_r(\theta_t + \mathbf{u} \cdot \nabla \theta) = \Delta \theta + Rw, \tag{5-2-3}$$

where $(\mathbf{x},\ t,\ \mathbf{u},\ p,\ T)$ have been made dimensionless with respect to $(h,\ h^2/\nu,\ \nu/h,\ \rho_0 \nu^2/h^2,\ (\frac{\beta \nu^3}{\alpha g \kappa h^2})^{1/2})$ respectively, h is fluid depth, ν is kinematic viscosity, ρ_0 is density, κ is thermal diffusivity, α is coefficient of thermal expansion, ϕ is porosity, $\mathbf{k} = (0, 0, 1)$ is the vertical unit vector, $\beta = (T_0 - T_1)/h$ is temperature gradient and $R = (\frac{\alpha \beta g h^4}{\kappa \nu})^{1/2}$, $T = \frac{2h^2\Omega}{\nu}$, $P_r = \frac{\nu}{\kappa}$, $P_0 = \frac{h^2}{k}$.

Equation (5-2-1) is called the Darcy-Brinkman-Boussinesq equation involving convective inertia, usual viscous force, as well as Darcy resistance force and Boussinesq approximation in the body force.

For the boundary conditions for the functions $\mathbf{u}$, θ, we suppose that $\mathbf{u}$, θ, p are periodic in x, y with period $2a_1, 2a_2$, respectively, and are stress free at the surfaces

$z = 0, 1$ so that

$$u_{,z} = v_{,z} = w = \theta = 0\,, \qquad z = 0, 1, \tag{5-2-4}$$

where $\mathbf{u} = (u, v, w)$ and $u_{,z} = \partial u/\partial z$.

To exclude rigid motions we assume that the mean values of u, v are zero, i.e., we require

$$\int_\Omega u\,dx = \int_\Omega v\,dx = 0\,, \tag{5-2-5}$$

where $\Omega = (\,0,\, 2a_1\,] \times (\,0,\, 2a_2\,] \times (\,0,\, 1\,)$.

3. An Energy Functional and Related Evolution Equation.

Let u be any of the four components of the vector of the perturbation $(\,\mathbf{u},\, \theta\,)$. The Hilbert space H coincides with $\mathbf{J}(\Omega) \times L^2(\Omega)$, and $\mathbf{J}(\Omega)$ is the subspace of $\mathbf{L}^2(\Omega)$ consisting of solenoidal vector functions $\mathbf{u}$ with $w = 0$ at $z = 0, 1$.

The problem described by (5-2-1)-(5-2-5) can be written as an evolution equation of the form

$$Bu_t = Au + RSu + TMu + Nu$$

in the Hilbert space H, in which B is a positive diagonal matrix, A is a linear differential operator, S is a bounded symmetic operator, M is a skew-symmetric operator , N is a nonlinear operator, and R and T are two dimensionless parameters. Following Galdi and Padula (1990) we can conclude that this system is a weakly coupled system. Besides the usual energy term, an appropriate energy functional is then proposed by considering the interaction between S and M. Rather than citing some abstract theorems and verifying all the hypotheses for weakly coupled systems, here we prefer to derive the evolution equation more directly. The procedure followed below, of course, draws on the basic idea from Galdi and Padula (1990).

Let Π be a projector operator from $\mathbf{L}^2(\Omega)$ onto $\mathbf{J}(\Omega)$; the following properties hold:

$$\begin{aligned} &1)\quad \Pi\mathbf{v} = \mathbf{v}, &&\text{if } \mathbf{v} \in \mathbf{J}(\Omega), \\ &2)\quad \Pi\mathbf{v} = \mathbf{v} - \nabla\psi, &&\text{if } \mathbf{v} \in \mathbf{L}^2(\Omega) \setminus \mathbf{J}(\Omega), \end{aligned} \tag{5-3-1}$$

with ψ (periodic in x, y, depending on $\mathbf{v}$) satisfying the problem

$$\begin{aligned} &\Delta\psi = \nabla \cdot \mathbf{v} \\ &\psi_{,z} = \mathbf{v} \cdot \mathbf{k} \quad \text{at} \quad z = 1, 0. \end{aligned} \tag{5-3-2}$$

For $(\mathbf{u}, \theta) \in (\mathbf{C}^2(\bar{\Omega}) \times C^2(\bar{\Omega})) \cap (\mathbf{J}(\Omega) \times L^2(\Omega))$, multiplying (5-2-1) by $\mathbf{u}$, (5-2-3) by θ, integrating over Ω, and using (5-2-4), we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 = -\|\nabla\mathbf{u}\|^2 - P_0\|u\|^2 + R < \theta w >, \tag{5-3-3}$$

$$\frac{1}{2}P_r\frac{d}{dt}\|\theta\|^2 = -\|\nabla\theta\|^2 + R < \theta w >, \tag{5-3-4}$$

where $\|\cdot\|$ denotes $\mathbf{L}^2(\Omega)$ (or $L^2(\Omega)$) norm, $< \cdot >$ denotes the integral over Ω.

Multiplying (5-2-1) by $\Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k}))$, and subsequently by $\Pi(\mathbf{k} \times \Pi(\theta\mathbf{k}))$ respectively, integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Pi(\mathbf{u} \times \mathbf{k})\|^2 = &- \|\mathbf{u}_{,z}\|^2 - P_0\|\Pi(\mathbf{u} \times \mathbf{k})\|^2 + R < \theta\mathbf{k} \cdot \Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k})) > \\ &- \frac{1}{\phi} < (\mathbf{u} \cdot \nabla\mathbf{u})\mathbf{u} \cdot \Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k})) >, \end{aligned} \tag{5-3-5}$$

$$\begin{aligned} < \Pi(\mathbf{u} \times \mathbf{k})_t \cdot (\theta\mathbf{k}) >= &- < \mathrm{curl}(\mathbf{u} \times \mathbf{k}) \cdot \mathrm{curl}(\theta\mathbf{k}) > - P_0 < \Pi(\mathbf{u} \times \mathbf{k}) \cdot (\theta\mathbf{k}) > \\ &- T < (\mathbf{u} \times \mathbf{k}) \cdot \Pi(\mathbf{k} \times \Pi(\theta\mathbf{k})) > - \frac{1}{\phi} < (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \Pi(\mathbf{k} \times \Pi(\theta\mathbf{k})) >, \end{aligned} \tag{5-3-6}$$

where the properties in (5-3-1) and (5-3-2) are used to derive these equations.

Multiplying (5-2-3) by $\Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k}$ and integrating over Ω, we also have

$$\begin{aligned} P_r < \theta_t\Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k} >= &- < \mathrm{curl}(\mathbf{u} \times \mathbf{k}) \cdot \mathrm{curl}(\theta\mathbf{k}) > + R < \Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k}w > \\ &- P_r < (\mathbf{u} \cdot \nabla\theta)\Pi(\mathbf{u} \times \mathbf{k}) \cdot \mathbf{k} >. \end{aligned} \tag{5-3-7}$$

Finally, multiplying (5-2-1) by $\Delta\mathbf{u}$, (5-2-3) by $\Delta\theta$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 = -\|\Delta\mathbf{u}\|^2 - P_0\|\nabla\mathbf{u}\|^2 - R < \Delta w\theta > + \frac{1}{\phi} < (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u} >, \quad (5\text{-}3\text{-}8)$$

$$\frac{1}{2}P_r\frac{d}{dt}\|\nabla\theta\|^2 = -\|\Delta\theta\|^2 - R < \Delta\theta w > - P_r < \mathbf{u}\cdot\nabla\theta\Delta\theta > . \quad (5\text{-}3\text{-}9)$$

Then we choose the same energy functional as constructed by Galdi and Padula (1990, p.246, formulation (11.8))

$$E = \frac{1}{2}(\|\mathbf{u}\|^2 + P_r\|\theta\|^2 + \lambda_2\|\Pi(\mathbf{u}\times\mathbf{k})\|^2 + 2\lambda P_r^{\frac{1}{2}} < \Pi(\mathbf{u}\times\mathbf{k})\cdot\theta\mathbf{k} >), \quad (5\text{-}3\text{-}10)$$

$$E_1 = \frac{1}{2}(\|\mathbf{u}\|^2 + P_r\|\theta\|^2 + \|\nabla\mathbf{u}\|^2 + P_r\|\nabla\theta\|^2), \quad (5\text{-}3\text{-}11)$$

and

$$V = E + \eta E_1 \quad (5\text{-}3\text{-}12)$$

where λ, λ_2, η are the positive coupling parameters. To ensure that E is positive-definite, the condition

$$\lambda_2 > \lambda^2 \quad (5\text{-}3\text{-}13)$$

must be imposed. Using expressions (5-3-10) to (5-3-12), together with all the results in (5-3-3) to (5-3-9), we arrive at

$$\frac{dV}{dt} = F(u) - D(u) + N(u) + \eta(F_1(u) + D_1(u) + N_1(u)), \quad (5\text{-}3\text{-}14)$$

where

$$
\begin{aligned}
F(u) = 2R < \theta w > &- \lambda P_r^{1/2}(1+1/P_r) < \mathrm{curl}(\mathbf{u}\times\mathbf{k})\cdot\mathrm{curl}(\theta\mathbf{k}) > \\
&- \lambda P_0 P_r^{1/2} < \Pi(\mathbf{u}\times\mathbf{k})\cdot\theta\mathbf{k} > + \lambda\frac{R}{P_r^{1/2}} < \Pi(\mathbf{u}\times\mathbf{k})\cdot\mathbf{k}w > \\
&- \lambda P_r^{1/2} T < (\mathbf{u}\times\mathbf{k})\cdot\Pi(\mathbf{k}\times\Pi(\theta\mathbf{k})) > \\
&+ \lambda_2 R < \theta\mathbf{k}\cdot\Pi(\mathbf{k}\times\Pi(\mathbf{u}\times\mathbf{k})) >, \qquad (5\text{-}3\text{-}15)
\end{aligned}
$$

$$
D(u) = \|\nabla\mathbf{u}\|^2 + \|\nabla\theta\|^2 + \lambda_2\|\mathbf{u}_{,z}\|^2 + P_0\|\mathbf{u}\|^2 + \lambda_2 P_0\|\Pi(\mathbf{u}\times\mathbf{k})\|^2, \qquad (5\text{-}3\text{-}16)
$$

$$
\begin{aligned}
N(u) = -\frac{\lambda_2}{\phi} < (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Pi(\mathbf{k}\times\Pi(\mathbf{u}\times\mathbf{k}) > &- \frac{\lambda P_r^{1/2}}{\phi} < (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Pi(\mathbf{k}\times\Pi(\theta\mathbf{k})) > \\
&- \lambda P_r^{1/2} < (\mathbf{u}\cdot\nabla\theta)(\Pi(\mathbf{u}\times\mathbf{k})\cdot\mathbf{k}) >, \qquad (5\text{-}3\text{-}17)
\end{aligned}
$$

$$
F_1(u) = 2R < \nabla\theta\cdot\nabla w > + 2R < \theta w >, \qquad (5\text{-}3\text{-}18)
$$

$$
D_1(u) = \|\Delta\mathbf{u}\|^2 + \|\Delta\theta\|^2 + \|\nabla\mathbf{u}\|^2 + \|\nabla\theta\|^2 + P_0\|\nabla\mathbf{u}\|^2 + P_0\|\mathbf{u}\|^2, \qquad (5\text{-}3\text{-}19)
$$

$$
N_1(u) = -\frac{1}{\phi} < (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u} > - P_r < (\mathbf{u}\cdot\nabla\theta)\Delta\theta > . \qquad (5\text{-}3\text{-}20)
$$

On using the properties of operator Π in (5-3-1) and (5-3-2), we can write

$$
\Pi(\mathbf{k}\times\Pi(\mathbf{u}\times\mathbf{k})) = \mathbf{k}\times(\mathbf{u}\times\mathbf{k}) - \mathbf{k}\times\nabla\zeta - \nabla\zeta_1 \qquad (5\text{-}3\text{-}21)
$$

with

$$
\begin{aligned}
&\Delta\zeta = \nabla\cdot(\mathbf{u}\times\mathbf{k}) = v_{,x} - u_{,y} \\
&\zeta_{,z} = 0 \qquad \text{at} \quad z = 0, 1 \qquad (5\text{-}3\text{-}22)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta\zeta_1 = \nabla\cdot(\mathbf{k}\times(\mathbf{u}\times\mathbf{k}) - \mathbf{k}\times\nabla\zeta) = u_{,x} + v_{,y} \\
&\zeta_{1,z} = 0 \qquad \text{at} \quad z = 0, 1. \qquad (5\text{-}3\text{-}23)
\end{aligned}
$$

Also we write

$$\Pi(\mathbf{k} \times \Pi(\theta\mathbf{k})) = \mathbf{k} \times \Pi(\theta\mathbf{k}) = \mathbf{k} \times (\theta\mathbf{k} - \nabla\tau) = -(\mathbf{k} \times \nabla)\tau \tag{5-3-24}$$

with

$$\Delta\tau = \theta_{,z}$$
$$\tau_{,z} = 0 \quad \text{at} \quad z = 0, 1 \tag{5-3-25}$$

Then we observe that

$$< \theta\mathbf{k} \cdot \Pi(\mathbf{k} \times \Pi(\mathbf{u} \times \mathbf{k})) >= - < \theta\zeta_{1,z} >=< \theta_{,z}\zeta_1 >, \tag{5-3-26}$$

and

$$< (\mathbf{u} \times \mathbf{k}) \cdot \Pi(\mathbf{k} \times \Pi(\theta\mathbf{k})) > = - < (v\tau_{,y} + u\tau_{,x}) >=< \tau(u_{,x} + v_{,y}) >$$
$$=< \tau\Delta\zeta_1 >=< (\Delta\tau)\zeta_1 >=< \theta_{,z}\zeta_1 > . \tag{5-3-27}$$

With the choice

$$\lambda_2 = \lambda T P_r^{1/2}/R \tag{5-3-28}$$

and using (5-3-26) and (5-3-27), we eliminate the last two terms in (5-3-15), which in general are not of one sign and may destablize the solution. Considering the condition (5-3-13), we are led to choose

$$\lambda = \xi T P_r^{1/2}/R, \qquad \xi \in (0,1). \tag{5-3-29}$$

4. A Priori Estimation of $\dot{V}$.

Now we need an a priori estimate of $\dot{V}$ to find the sufficient conditions which ensure that the energy V decays monotonically to zero.

We make the following observations

$$
\begin{aligned}
N(u) &\le \frac{\lambda_2}{\phi}\sup_\Omega |u|\|\nabla \mathbf{u}\|\|\Pi(\mathbf{u}\times\mathbf{k})\| + \lambda P_r^{1/2}\sup_\Omega |u|\|\nabla\theta\|\|\Pi(\mathbf{u}\times\mathbf{k})\| \\
&\quad + \frac{\lambda P_r^{1/2}}{\phi}\sup_\Omega |u|\|\nabla \mathbf{u}\|\|\Pi(\theta\mathbf{k})\| \\
&\le \frac{\lambda_2 c}{\phi}\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\|\|\Pi(\mathbf{u}\times\mathbf{k})\| + \frac{\lambda c}{\phi}P_r^{1/2}\|\Delta\mathbf{u}\|\|\nabla\mathbf{u}\|\|\theta\| \\
&\quad + \lambda c P_r^{1/2}\|\Delta\mathbf{u}\|\|\nabla\theta\|\|\Pi(\mathbf{u}\times\mathbf{k})\| \\
&\le \frac{c\sqrt{\lambda_2}}{\phi\sqrt{2}}D_1E^{1/2} + \frac{\lambda c}{\phi\sqrt{2}}D_1E^{1/2} + \frac{\lambda c p_r^{1/2}}{(2\lambda_2)^{1/2}}D_1E^{1/2} \\
&= bD_1E^{1/2}, \qquad (5\text{-}4\text{-}1)
\end{aligned}
$$

where

$$
b = \frac{c}{\phi}\sqrt{\frac{\lambda_2}{2}} + \frac{\lambda c}{\phi\sqrt{2}} + \frac{\lambda c P_r^{1/2}}{(2\lambda_2)^{1/2}} \qquad (5\text{-}4\text{-}2)
$$

and

$$
\sup_\Omega |u| \le c\,\|\Delta\mathbf{u}\|, \qquad (5\text{-}4\text{-}3)
$$

$$
c = \frac{3^{1/2}}{2^{1/4}[\pi h^3(\sqrt{2}-1)]^{1/3}} + \frac{40h^{3/5}}{3}, \qquad (5\text{-}4\text{-}4)
$$

with

$$
h = \min\{a_1, a_2, 1\}
$$

(cf. Galdi and Straughan 1985, p. 267). We also observe

$$
\begin{aligned}
N_1(u) &\le \sup_\Omega |u|\left(\frac{1}{\phi}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\| + P_r\|\nabla\theta\|\|\Delta\theta\|\right) \\
&\le \sqrt{2}c\left(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2}\right)E_1^{1/2}D_1, \qquad (5\text{-}4\text{-}5)
\end{aligned}
$$

and

$$
\begin{aligned}
F_1(u) &\le 2R\|w\|\|\Delta\theta\| + 2R\|w\|\|\theta\| \\
&\le \frac{2}{\pi}R\|w_{,z}\|(\|\Delta\theta\| + \frac{1}{\pi}\|\nabla\theta\|) \\
&\le \frac{1}{2}(\|\nabla\theta\|^2 + \|\Delta\theta\|^2) + 2R^2\frac{\pi^2+1}{\pi^4}\|\mathbf{u}_{,z}\|^2 \\
&\le \frac{1}{2}D_1 + \frac{2R^2(\pi^2+1)}{\lambda_2\pi^4}D, \qquad (5\text{-}4\text{-}6a)
\end{aligned}
$$

or

$$
\begin{aligned}
F_1(u) &\le \frac{2R}{\pi}\|\nabla w\|(\|\Delta\theta\| + \frac{1}{\pi}\|\nabla\theta\|) \\
&\le \frac{1}{2}D_1 + \frac{2R^2(1+\pi^2)}{\pi^4}D, \qquad (5\text{-}4\text{-}6b)
\end{aligned}
$$

where the Poincaré inequalities have been used in the derivation of above results.

Define

$$
m = \max_{\varphi}\frac{F}{D}, \qquad (5\text{-}4\text{-}7)
$$

where φ is a set of the admissible functions,

$$
\varphi = \{\mathbf{u}, \theta \text{ which are periodic in } x \text{ and } y \text{ and satisfy (5-2-1) to (5-2-4) }\}. \qquad (5\text{-}4\text{-}8)
$$

We suppose first that $T > 1$; using all the results (5-4-1) to (5-4-6a) and (5-4-7) in (5-3-13) yields

$$
\begin{aligned}
\dot{V} \le& \{(m-1) + \frac{2\eta R^2(\pi^2+1)}{\lambda_2\pi^4}\}D - \frac{1}{2}\eta D_1 \\
&+ b\bar{D}_1 E^{1/2} + \sqrt{2}\eta c(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2})E_1^{1/2}D_1. \qquad (5\text{-}4\text{-}9)
\end{aligned}
$$

Choosing

$$
\eta = \frac{\lambda_2\pi^4(1-m)}{4R^2(\pi^2+1)} \qquad (5\text{-}4\text{-}10)
$$

and setting

$$\mathcal{D} = \frac{1}{2}[(1-m)D + \eta D_1]. \tag{5-4-11}$$

we conclude that

$$\dot{V} \leq -\mathcal{D}(1 - AV^{1/2}), \tag{5-4-12}$$

where

$$A = 2\frac{b}{\eta} + \frac{2\sqrt{2}c(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2})}{\eta^{1/2}}. \tag{5-4-13}$$

By virtue of the Poincaré (Writinger) inequality (Galdi and Straughan, 1985), we also have

$$\mathcal{D} \geq \frac{(1-m)}{2P^*}V \tag{5-4-14}$$

where

$$\begin{aligned} P^* &= 1 \qquad if \;\; P_r \leq 1 \\ &= P_r \quad\; if \;\; P_r > 1 \end{aligned} \tag{5-4-15}$$

Then from (5-4-12) we may show that

$$V(t) \leq V(0)\exp\{-\frac{\pi^2(1-m)}{2P^*}[1 - AV^{1/2}(0)]t\} \tag{5-4-16}$$

which means that $V(t) \to 0$ in a monotonic fashion as $t \to \infty$, whenever

$$V(0) < A^{-2}. \tag{5-4-17}$$

If $T \leq 1$, we use (5-4-6b) to replace (5-4-6a), then get the estimation in the form

$$\begin{aligned} \dot{V} \leq \{(m-1) + \frac{2\eta R^2(\pi^2+1)}{\pi^4}\}D - \frac{1}{2}\eta D_1 \\ + b_1 D_1 E^{1/2} + \sqrt{2}\eta c(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2})E_1^{1/2} D_1, \end{aligned} \tag{5-4-18}$$

where b_1 coincides with the value of b for $T = 1$. From (5-4-18), it follows that (5-4-16) holds with

$$A = \frac{2b_1}{\eta} + \frac{2\sqrt{2}c(\frac{1}{\phi} + \frac{1}{2}P_r^{1/2})}{\eta^{1/2}} \tag{5-4-19}$$

and

$$\eta = \frac{\pi^4(1-m)}{4R^2(1+\pi^2)}. \tag{5-4-20}$$

When $V(0) < A^{-2}$, we again obtain monotonic stability.

5. Variational Problem.

Now we solve the variational problem associated with the maximum problem (5-4-7). The corresponding Euler-Lagrange equations are

$$\begin{aligned} &m\Delta\mathbf{u} - mP_0\mathbf{u} + m\lambda_2\mathbf{u}_{,zz} - \lambda_2 P_0(\mathbf{k}\times\Pi(\mathbf{u}\times\mathbf{k})) + R\theta\mathbf{k} \\ &- \frac{\lambda(1+P_r)}{2\sqrt{P_r}}\mathbf{k}\times\text{curlcurl}(\theta\mathbf{k}) - \frac{\lambda P_0\sqrt{P_r}}{2}\mathbf{k}\times\Pi(\theta\mathbf{k}) \\ &+ \frac{\lambda R}{2\sqrt{P_r}}(\Pi(\mathbf{u}\times\mathbf{k})\cdot\mathbf{k})\mathbf{k} - \mathbf{k}\times\Pi(\mathbf{k}w)) = \nabla p, \end{aligned} \tag{5-5-1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{5-5-2}$$

$$m\Delta\theta + Rw - \frac{\lambda(1+P_r)}{2\sqrt{P_r}}\text{curlcurl}(\mathbf{u}\times\mathbf{k})\cdot\mathbf{k} - \frac{\lambda}{2}P_0\sqrt{P_r}\Pi(\mathbf{u}\times\mathbf{k})\cdot\mathbf{k} = 0. \tag{5-5-3}$$

Using the variables defined by (5-3-22) and (5-3-25) for ζ, τ respectively, and a variable ψ which is associated with $\Pi(\mathbf{k}w)$ in the form of

$$\Pi(\mathbf{k}w) = \mathbf{k}w - \nabla\psi$$

with

$$\begin{aligned} &\Delta\psi = w_{,z}, \\ &\psi_{,z} = 0 \quad \text{at} \quad z = 0, 1, \end{aligned} \tag{5-5-4}$$

we can write (5-5-1) and (5-5-3) as

$$
\begin{aligned}
&m\Delta \mathbf{u} - mP_0\mathbf{u} + m\lambda\frac{T\sqrt{P_r}}{R}\mathbf{u}_{,zz} - m\lambda\frac{T\sqrt{P_r}}{R}P_0(\mathbf{u} - w\mathbf{k} - (\mathbf{k}\times\nabla)\zeta) \\
&\quad + R\theta\mathbf{k} + \lambda\frac{(1+P_r)}{2\sqrt{P_r}}(\nabla\theta_{,z}\times\mathbf{k}) + \frac{\lambda}{2}P_0\sqrt{P_r}(\mathbf{k}\times\nabla\tau) \\
&\quad + \frac{\lambda R}{2\sqrt{P_r}}(\zeta_{,z}\mathbf{k} - \mathbf{k}\psi) = \nabla p, \qquad (5\text{-}5\text{-}5)
\end{aligned}
$$

$$
m\Delta\theta + Rw - \frac{\lambda(1+P_r)}{2\sqrt{P_r}}\omega_{,z} + \frac{\lambda}{2}P_0\sqrt{P_r}\zeta_{,z} = 0, \qquad (5\text{-}5\text{-}6)
$$

where $\omega = (\nabla\times\mathbf{u})\cdot\mathbf{k}$. Operating on (5-5-5) with $\mathbf{k}\cdot\Delta$curl and then with $\mathbf{k}\cdot\Delta$curlcurl , operating on (5-5-6) with Δ, and using (5-3-27), we arrive at

$$
\begin{aligned}
&m\Delta^2\omega - mP_0\Delta\omega + m\frac{\lambda T\sqrt{P_r}}{R}\Delta\omega_{,zz} - m\frac{\lambda T\sqrt{P_r}P_0}{R}\omega_{,zz} \\
&\quad - \lambda\frac{(1+P_r)}{2\sqrt{P_r}}\Delta\Delta_1\theta_{,z} + \frac{\lambda P_0\sqrt{P_r}}{2}\Delta_1\theta_{,z} - \frac{\lambda R}{2\sqrt{P_r}}\Delta_1 w_{,z} = 0,
\end{aligned}
\qquad (5\text{-}5\text{-}7)
$$

$$
\begin{aligned}
&m\Delta^3 w - mP_0\Delta^2 w + m\frac{\lambda T\sqrt{P_r}}{R}\Delta^2 w_{,zz} - m\frac{\lambda T\sqrt{P_r}P_0}{R}\Delta^2 w \\
&\quad + R\Delta\Delta_1\theta + \frac{\lambda R}{2\sqrt{P_r}}\Delta_1\omega_{,z} = 0, \qquad (5\text{-}5\text{-}8)
\end{aligned}
$$

$$
m\Delta^2\theta + R\Delta w - \frac{\lambda(1+P_r)}{2\sqrt{P_r}}\Delta\omega_{,z} + \frac{\lambda}{2}P_0\sqrt{P_r}\omega_{,z} = 0. \qquad (5\text{-}5\text{-}9)
$$

Eliminating variables θ and ω yields

$$
\begin{aligned}
&\Delta^2[(\Delta - P_0) + \lambda T\sqrt{P_r}/R(D^2 - P_0)]\cdot \\
&\{(\frac{\lambda(1+P_r)}{2\sqrt{P_r}}\Delta - \frac{\lambda P_0\sqrt{P_r}}{2})^2\Delta_1 D^2 - m^2\Delta^2(\Delta - P_0)(\Delta + \frac{\lambda T\sqrt{P_r}}{R}D^2)\}w \\
&+ \{-\frac{\lambda^2 R^2}{4P_r}\Delta^2\Delta_1^2 D^2 + mR^2\Delta^2\Delta_1(\Delta - P_0)(\Delta + \frac{\lambda T\sqrt{P_r}}{R}D^2)\}w = 0,
\end{aligned}
\qquad (5\text{-}5\text{-}10)
$$

where $D = \partial/\partial z$, $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

On employing standard normal mode analysis we may write $w = W(z)e^{2i(a_1x+a_2y)}$. The boundary conditions (5-2-4) together with the equations (5-2-1) to (5-2-3) can produce $D^{(2l)}W = 0$ at $z = 0, 1$ for $l = 1, 2, \cdots$. This indicates that W(z) can be assumed to have the form $W(z) = H \sin r\pi z$, for $r = 1, 2, \cdots$ and H an arbitrary constant. Substituting these functions into equation (5-5-10) and noting (5-3-29), we obtain

$$
\begin{aligned}
&-(r^2\pi^2+a^2)^2[(r^2\pi^2+a^2+P_0)+\frac{\xi T^2 P_r}{R^2}(r^2\pi^2+P_0)]\times \\
&\{\frac{\xi^2T^2}{4R^2}[(1+P_r)(r^2\pi^2+a^2)+P_0P_r]^2 r^2\pi^2a^2 \\
&-m^2(r^2\pi^2+a^2)^2(r^2\pi^2+a^2+P_0)(r^2\pi^2+a^2+\frac{\xi T^2P_r}{R^2}r^2\pi^2)\} \\
&+\{\frac{\xi^2T^2}{4}(r^2\pi^2+a^2)^2r^2\pi^2a^4-mR^2a^2(r^2\pi^2+a^2)^2(r^2\pi^2+a^2+P_0)\times \\
&(r^2\pi^2+a^2+\frac{\xi T^2P_r}{R^2}r^2\pi^2)\}=0, \qquad (5\text{-}5\text{-}11)
\end{aligned}
$$

where $a^2 = 4(a_1^2 + a_2^2)$.

For convenience, we rewrite the above equation as a polynomial equation in R^2 in the form

$$p(R^2, m) = R^6 + A_1(m)R^4 + B_1(m)R^2 + C_1(m) = 0, \qquad (5\text{-}5\text{-}12)$$

where

$$A_1(m) = -m^2 r^4 \pi^4 \frac{(1+\alpha)^2(1+\alpha+\beta)}{\alpha} + \frac{\xi T^2 P_r}{(1+\alpha)}$$

$$- \frac{\xi^2 T^2}{4} \frac{\alpha}{(1+\alpha)(1+\alpha+\beta)}, \qquad (5\text{-}5\text{-}13)$$

$$B_1(m) = r^4 \pi^4 \{ -m^2 \xi T^2 P_r \frac{(1+\alpha)(1+\alpha+\beta)}{\alpha}$$

$$- m^2 \xi T^2 P_r \frac{(1+\alpha)^2(1+\beta)}{\alpha}$$

$$+ \frac{\xi^2 T^2}{4} \frac{[(1+P_r)(1+\alpha)+\beta P_r]^2}{(1+\alpha)} \}, \qquad (5\text{-}5\text{-}14)$$

$$C_1(m) = r^4 \pi^4 \{ -m^2 \xi^2 T^4 P_r^2 \frac{(1+\alpha)(1+\beta)}{\alpha}$$

$$+ \frac{\xi^3 T^4 P_r}{4} \frac{(1+\beta)[(1+P_r)(1+\alpha)+\beta P_r]^2}{(1+\alpha)(1+\alpha+\beta)} \}, \qquad (5\text{-}5\text{-}15)$$

with

$$\alpha = \frac{a^2}{r^2 \pi^2}, \qquad \beta = \frac{P_0}{r^2 \pi^2}. \qquad (5\text{-}5\text{-}16)$$

Choosing

$$\xi < \frac{4 P_r (1+\alpha)^2 (1+\alpha+\beta)}{\alpha[(1+P_r)(1+\alpha)+\beta P_r]^2}, \qquad (5\text{-}5\text{-}17)$$

and denoting $A = A_1(1),\ B = B_1(1),\ C = C_1(1)$, we observe that there is one and only one positive number R_c^2 satisfying the equation

$$P(R^2, 1) = R^6 + AR^4 + BR^2 + C = 0, \qquad (5\text{-}5\text{-}18)$$

and

$$P(R^2, 1) < 0 \quad \text{for any} \quad R^2 < R_c^2, \qquad (5\text{-}5\text{-}19)$$

because both B and C are negative under the assumption of (5-5-17).

Moreover, a straightforward calculation shows that

$$\frac{dA_1(m)}{dm} < A, \quad \frac{dB_1(m)}{dm} < B, \quad \frac{dC_1(m)}{dm} < C, \qquad (5\text{-}5\text{-}20)$$

for any $m \geq 1$.

Therefore a comparison shows for any $R^2 < R_c^2$

$$\frac{dP(R^2,m)}{dm} < P(R^2,1) < 0, \qquad (5\text{-}5\text{-}21)$$

for any $m \geq 1$. This implies that $P(R^2,m) < 0$ if $m \geq 1$ for any $R^2 < R_c^2$, while R_c^2 is a positive root of equation (5-5-12) at the critical argument $m = 1$. We thus conclude that $R < R_c$ implies $m < 1$. We also remark here that (5-5-17) will certainly be satisfied if the further choice

$$\xi < \frac{4P_r(1+\alpha)}{\alpha(1+P_r)^2(1+\beta)} \qquad (5\text{-}5\text{-}22)$$

is required.

With this choice, it is also straightforward to verify that $r = 1$ minimizes R_c. Actually taking differentiation with respect to r^2 on both sides of (5-5-18), we reduce the equation to

$$(3R_c^4 + 2AR_c^2 + B)R_c^{2'} = -A'R_c^4 - B'R_c^2 - C', \qquad (5\text{-}5\text{-}23)$$

where prime denotes the differentiation with respect to r^2. An inspection of expressions for A, B, C and the following observations

$$\begin{aligned}
A' &= \frac{\partial A}{\partial r^2} + \frac{\partial A}{\partial \beta}\frac{\partial \beta}{\partial r^2} = \frac{\partial A}{\partial r^2} + \frac{\partial A}{\partial \beta}(-\frac{\beta}{r^2}) \\
&= -2r^2\pi^4\frac{(1+\alpha)^2(1+\alpha+\beta)}{\alpha} + r^2\pi^4\frac{(1+\alpha)^2\beta}{\alpha} - \frac{\xi^2T^2\alpha\beta}{(1+\alpha)(1+\alpha+\beta)^2r^2} \\
&= -r^2\pi^4\frac{(1+\alpha)^2(1+\alpha+\beta)}{\alpha} - r^2\pi^4\frac{(1+\alpha)^3}{\alpha} - \frac{\xi^2T^2\alpha\beta}{(1+\alpha)(1+\alpha+\beta)^2r^2} < 0, \\
B' &= \frac{\partial B}{\partial r^2} + \frac{\partial B}{\partial \beta}(-\frac{\beta}{r^2}) \\
&= r^2\pi^4\{-\xi T^2P_r\frac{(1+\alpha)(1+\alpha+\beta)}{\alpha} - \xi T^2P_r\frac{(1+\alpha)^2(1+\beta)}{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+\xi^2T^2\frac{[(1+P_r)(1+\alpha)+\beta P_r]^2}{4(1+\alpha)}\} \\
&+r^2\pi^4\{-2\xi T^2P_r\frac{(1+\alpha)^2}{\alpha}+\frac{\xi^2T^2}{4(1+\alpha)}[(1+P_r)^2(1+\alpha)^2-\beta^2P_r^2]\},
\end{aligned}
$$

$$
\begin{aligned}
C' &= \frac{\partial C}{\partial r^2}+\frac{\partial C}{\partial \beta}(-\frac{\beta}{r^2}) \\
&= r^2\pi^4\{-\xi^2T^4P_r^2\frac{(1+\alpha)(1+\beta)}{\alpha}+\frac{\xi^3T^4P_r}{4}\frac{(1+\beta)[(1+P_r)(1+\alpha)+\beta P_r]^2}{(1+\alpha)(1+\alpha+\beta)}\} \\
&\quad +r^2\pi^4\{-\xi^2T^4P_r^2\frac{(1+\alpha)}{\alpha}++\frac{\xi^3T^4P_r}{4}(1+\beta)(1+P_r)^2\} \\
&\quad +r^2\pi^4\frac{\xi^3T^4P_r}{4}\{-\frac{(1+\beta)(1+P_r)^2\beta}{(1+\alpha+\beta)}-\frac{(1+\beta)P_r^2\beta^2}{(1+\alpha)(1+\alpha+\beta)} \\
&\quad -\frac{[(1+P_r)(1+\alpha)+\beta P_r]\alpha\beta}{(1+\alpha)(1+\alpha+\beta)^2}\}
\end{aligned}
\tag{5-5-24}
$$

together with the restriction of the choice in (5-5-22) show that the right hand side of (5-5-23) is positive while the term in the brackets on the left side is positive. We thus conclude that $\frac{\partial R_c^2}{\partial (r^2)} > 0$ and the minimum is attained at $r = 1$. The critical energy bound will be defined by

$$
R_e^2 = \min_{\alpha}\max_{\xi} R_c(\alpha,\xi,P_0,P_r,T^2) \tag{5-5-25}
$$

6. Linear Instability Bound.

In order to make a comparison, in this section we briefly discuss the linear instability of system (5-2-1) to (5-2-5).

By standard manipulations, we take curl and curlcurl operators on the linearized version of equation (5-2-1) and then consider the third components of the resulting equations. The results read

$$
\omega_t = \Delta\omega - P_0\omega + TDw\,, \tag{5-6-1}
$$

$$
\Delta w_t = \Delta^2 w - P_0\Delta w + R\Delta_1\theta - TD\omega. \tag{5-6-2}
$$

where $\omega = (\nabla\times\mathbf{u})\cdot\mathbf{k}$, $w = \mathbf{u}\cdot\mathbf{k}$, $D = \partial/\partial z$.

The linearized form of equation (5-2-3) gives

$$P_r\theta_t = \Delta\theta + Rw. \tag{5-6-3}$$

As is customary, we use the form $w = W(\mathbf{x})e^{\sigma t}$ with similar expressions for θ and ω. Eliminating ω and θ between (5-6-1) and (5-6-3) leads to

$$\begin{aligned} R^2(\Delta - P_0 - \sigma)\Delta_1 W = (\Delta - P_r\sigma)(\Delta - P_0 - \sigma)^2\Delta W \\ + T^2(\Delta - P_r\sigma)D^2W. \end{aligned} \tag{5-6-4}$$

The boundary condition (5-2-5) together with the linearized version of equations (5-2-1), (5-2-3) and normal mode necessarily suggests $W = H\sin r\pi z e^{i(2a_1x+2a_2y)}$, for $r = 1, 2, \cdots$, and H is an arbitrary constant. Substituting these functions into (5-6-4), we obtain

$$\begin{aligned} &R^2(r^2\pi^2 + a^2 + P_0 + \sigma)a^2 \\ &= (r^2\pi^2 + a^2 + P_0 + \sigma)^2(r^2\pi^2 + a^2 + P_r\sigma)(r^2\pi^2 + a^2) \\ &\quad + T^2(r^2\pi^2 + a^2 + P_r\sigma)r^2\pi^2. \end{aligned} \tag{5-6-5}$$

For the onset of stationary convection, setting $\sigma = 0$ in (5-6-5) we obtain

$$R^2 = \frac{(r^2\pi^2 + a^2)^2(r^2\pi^2 + a^2 + P_0)^2 + T^2r^2\pi^2(r^2\pi^2 + a^2)}{(r^2\pi^2 + a^2 + P_0)a^2}. \tag{5-6-6}$$

A straighdforward calculation shows $\frac{\partial R^2}{\partial(r^2)} > 0$ and the minimum of R^2 is attained at $r = 1$. For the onset of oscillatory convection, setting $\sigma = i\sigma_1$ in (5-6-5), we deduce

$$\begin{aligned} R^2 = [(r^2\pi^2 + a^2 + P_0)(r^2\pi^2 + a^2) - P_r\sigma^2]\frac{(r^2 + a^2)}{a^2} \\ + \frac{T^2r^2\pi^2}{a^2}\frac{(r^2\pi^2 + a^2 + P_0)(r^2\pi^2 + a^2) + P_r\sigma_1^2)}{(r^2\pi^2 + a^2 + P_0{}^2 + \sigma_1^2)}, \end{aligned} \tag{5-6-7}$$

and

$$\sigma_1^2 = -(r^2\pi^2 + a^2 + P_0)$$
$$= \frac{((r^2\pi^2 + a^2) - P_r(r^2 + \pi^2 + a^2 + P_0)}{(r^2 + \pi^2 + a^2) + P_r(r^2\pi^2 + a^2 + P_0)} \frac{T^2 r^2 \pi^2}{(r^2\pi^2 + a^2)} \tag{5-6-8}$$

It follows from (5-6-8) that the overstability can occur only when

$$P_r < \frac{\pi^2 + a^2}{\pi^2 + a^2 + P_0}, \tag{5-6-9}$$

and

$$T^2 > \frac{[(\pi^2 + a^2) + P_r(\pi^2 + a^2 + P_r)](\pi^2 + a^2)(\pi^2 + a^2 + P_0)}{[(\pi^2 + a^2) - P_r(\pi^2 + a^2 + P_0)]\pi^2}. \tag{5-6-10}$$

The critical Rayleigh number for stationary convection and oscillatory convection will be determined as the minimum value when a^2 varies. The bound at the onset of instability can be chosen by comparison between two of them; whichever is the lower one.

7. Conclusion.

We give some numerical calculations for the energy and linear critical bounds. In Tables 1 and 2 several values of R_e^2 as a function of T and P_0 are given. As we are concerned with high porosity systems only, comparatively small values for P_0 are considered. Our calculations agree with the results given by Galdi and Padula (1990) when the parameter $P_0 = 0$. For a comparison, the linear critical bounds are also given in Table 3 . This case is applicable only to stationary convection as we chose $P_r > 1$. When the parameter $P_0 = 0$, our results agree with those of Chandrasekhar (1961). For the low porosity system when P_0 becomes very large, the coupling paremeter ξ has to be very small due to the choice of (5-5-22). In this case R_e^2 tend to $P_0\pi^2/4$ which is the critical value without rotation.

We note that the critical energy numbers depend on the Taylor number T and Darcy number P_0. As T increases, the critical energy and linear Rayleigh numbers increase. For low values of the Taylor number, the effect of porosity is to stabilize the system, that is, as P_0 increases so does the value of R_e^2. However, since we are not aware of any experimental information in rotating porous media, it is not possible for us to make any direct comparison.

In this chapter we have been able to predict plausible results for the rotating Bénard problem in porous media by considering the generalized energy functional. We point out that had we considered the classical energy function or if we had used the Darcy model, we would not have been able to predict any rotating effect.

Table 1. Energy Critical Rayleigh Numbers

$P_r = 6.8$

T^2	$P_0 = 0.0$ R_e^2	$P_0 = 0.1$ R_e^2	$P_0 = 1.0$ R_e^2	R_{G-P}^2
0	657.511	661.95	701.69	657.511
10	672.381	676.79	716.33	672.609
50	723.235	727.64	767.01	723.236
100	775.105	779.56	819.19	775.104
200	858.523	863.12	904.02	858.514
300	926.967	931.52	973.76	926.750
400	985.907	990.82	1034.35	1001.92
1000	1249.95	1255.46	1305.73	1249.83

R_{G-P}^2-results obtained by Galdi & Padula

Table 2. Energy Critical Rayleigh Numbers

$P_r = 200$

T^2	$P_0 = 0.0$ R_e^2	$P_0 = 0.1$ R_e^2	$P_0 = 1.0$ R_e^2	$P_0 = 1.0$ R_{G-P}^2
0	657.511	661.95	701.69	657.511
10	676.37	680.76	719.05	676.75
50	738.95	743.27	781.86	738.94
100	800.99	805.33	844.01	800.99
250	940.09	944.63	984.88	
500	1106.76	1111.68	1154.96	
1000	1347.78	1353.32	1401.91	1347.78
2000	1690.67	1697.22	1754.25	
5000	2369.71	2377.97	2452.98	

R_{G-P}^2-results obtained by Galdi & Padula

Table 3. Linear Critical Rayleigh Numbers

T^2	$P_0 = 0.0$ R_c^2	$P_0 = 0.1$ R_c^2	$P_0 = 1.0$ R_c^2	R_{Ch}^2
0	657.511	661.95	701.69	6.575×10^2
10	677.08	681.33	719.53	6.771×10^2
50	748.31	752.00	785.85	
100	826.29	829.51	858.89	
250	1019.51	1021.92	1044.18	
500	1274.57	1276.30	1292.43	1.275×10^3
1000	1676.12	1677.15	1686.89	1.676×10^3
2000	2299.36	2346.74	2302.47	
5000	3669.78	3669.03	3662.60	3.670×10^3

R_{Ch}^2-results obtained by Chandrasekhar

Chapter 6

Unsteady Flow of Power Law Fluids in Porous Media

1. Introduction.

In this final chapter we turn our attention to another non-Darcian model which has been used to describe non-Newtonian fluids properties in porous media. The model selected here is a power law fluid which accounts for variable viscosity. By using this model we extend a mathematical analysis for two typical initial-boundary value problems which correspond to well-test cases in the oil industry.

During the last three decades, a great deal of progress has been made on the mathematical analysis of flow through porous media. There are a number of mathematical papers on porous media problems in the literature, for example, Oleinik et al. (1958), Aroson (1969), Gilding and Peletier (1976), Van Duyn and Peletier (1982), Alikakos and Rostamian (1981), Warren and Root (1963) and references cited therein. Much of this work, however, has been concerned with flow of Newtonian fluids in a porous medium. In the production of heavy crude oils, rheological studies indicate that some of them are non-Newtonian fluids. Thus, using Newtonian displacing fluids is not quite satisfactory to interpret the oil displacement efficiency. As a result of increasing practical interest in oil reservoir engineering, the steady and unsteady flow of non-Newtonian fluids through porous media has recently received special attention.

According to the recent articles, Pascal and Pascal (1985), Pascal (1990), a basic equation for power law fluids in the absence of yield stress, in one dimensional flow, may be written as

$$(\frac{\partial P}{\partial x})^{(1-b)/b}\frac{\partial^2}{\partial x^2} = ba^2\frac{\partial P}{\partial t}, \tag{6-1-1}$$

where $0 < b < 1$, the coefficient a^2 is given by the relation $a^2 = (\frac{\mu}{k})^{1/b}(c_0\phi + c_p)$,

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX in which ϕ is the porosity of the medium, k is the permeability of the medium, μ is the viscosity of the oil, and c_0 and c_p are two constants. Once the pressure distributions are determined from the equation (6-1-1), the velocity distributions may be obtained from the modified Darcy's law and expressed as

$$u(x,t) = (-\frac{k}{\mu}\frac{\partial P}{\partial x})^{1/b}.$$

By a suitable change of time or distance scale one can reduce the constant coefficient in (6-1-1) to unity and obtain equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}((\frac{\partial P}{\partial x})^m), \tag{6-1-2}$$

where $m = 1/b > 1$. When $m = 1$ equation (6-1-2) is well known as the diffusion equation.

In this chapter, the investigation is conducted to discuss the solutions of initial-boundary value problems associated with equation (6-1-2), with a constant boundary value condition and a constant derivative boundary value condition on a semi-bounded line, i.e.,

$$\text{I} \quad \begin{aligned} &P_t = ((P_x)^m)_x && 0 < x < \infty, \quad t > 0, && (6\text{-}1\text{-}3)\\ &P(0,t) = P_w && t \geq 0, && (6\text{-}1\text{-}4)\\ &P(x,0) = P_0(x) && 0 \leq x < \infty, && (6\text{-}1\text{-}5) \end{aligned}$$

where P_w is a positive constant, which represents the pressure in a well, $P_0(x)$ is an initial pressure function, and

$$\text{II} \quad \begin{aligned} &P_t = ((P_x)^m)_x && 0 < x < \infty, 0 < t < T, && (6\text{-}1\text{-}6)\\ &P_x(0,t) = A && 0 \leq t \leq T, && (6\text{-}1\text{-}7)\\ &P(x,0) = P_0(x) && 0 \leq x < \infty, && (6\text{-}1\text{-}8) \end{aligned}$$

where A is a positive constant which represents the constant flow rate and T is a positive constant.

It is well known that the well-test analysis for a short period of time is used to get useful information of the geological parameters of an oil reservoir. For example, the flow rate variation in a well produced at a constant pressure, or the pressure variation in time in a well produced at a constant flow rate, recorded and interpreted according to the mathematical solution of the basic equation describing the flow through a porous medium, are two adequate techniques used now in the field for determining reservoir properties. Problem I and Problem II respectively correspond to those two cases.

The article by Pascal and Pascal (1985) also investigated problems which are analogous to our Problem I and Problem II. The generalized Boltzman similarity transformation was used to reduce the basic equation (6-1-1) to an ordinary differential equation, then the exact solutions were easily found with the different boundary conditions. But the Boltzman transformation could only be used to get an exact solution for a constant pressure in a well, in the case when the initial pressure is also a constant. Moreover, this transformation fails to solve the problem in the case of the constant flow rate. The purpose of the present study is to give a rigorous theoretical analysis for Problem I and Problem II.

Equation (6-1-2) is a quasilinear parabolic equation, which is degenerate when $P_x = 0$. Because of this degeneracy, P_{xx} may not be continuous at the point where $P_x = 0$. We shall follow Oleinik *et al.* (1958) to introduce a class of generalized solutions for our problems.

Let H and H_T be the two sets defined by

$$H = \{(x,t) \mid 0 < x < \infty,\ 0 < t < \infty\},$$

$$H_T = \{(x,t) \mid 0 < x < \infty,\ 0 < t < T\}.$$

Throughout this chapter, we make the following assumption on the initial function $P_0(x)$:

(A_1) $P_0(x)$ is a positive , non-decreasing and Lipschitz continuous function on a semi-straight line $x \geq 0$ and $P_0(x) = P_e =$ const., where $x \geq a$ for a fixed point $a > 0$.

Definition 1.1 Let $P_0(x)$ satisfy the assumption (A_1), $P_0(0) = P_w$ and $P_e > P_w$. A function $P(x,t)$ will be called a generalized solution of Problem I if

(i) $P(x,t)$ is a positive, bounded, continuous function in $\bar{H}$ and $P(0,t) = P_w$;

(ii) $P_x(x,t)$ is nonnegative, bounded and continuous function in H;

(iii) $((P_x)^m)_x \in L^\infty_{loc}(R^+ \times R^+)$;

(iv) $P(x,t)$ satisfies the identity

$$\int_0^\infty \int_0^\infty \{(P_x)^m \varphi_x - P\varphi_t\}\, dxdt = \int_0^\infty P_0(x)\varphi(x,0)\, dx,$$

for all $\varphi \in C^1(\bar{H})$, which vanish for $x = 0$ and for large values of x and t.

Definition 1.2 Let $P_0(x)$ satisfy the assumption (A_1), $P_0'(0) = A$ and $P_0(0) > 0$. A function $P(x,t)$ will be called a generalized solution of Problem II if there exists a T^* with $0 < T^* < \infty$, and for any $0 < T \leq T^*$ such that

(i) $P(x,t)$ is a non-negative, bounded and continuous function in $\bar{H}_T$;

(ii) $P_x(x,t)$ is nonnegative, bounded and continuous function in H_T;

(iii) $((P_x)^m)_x \in L^\infty_{loc}(R^+ \times (0,T))$;

(iv) $P(x,t)$ satisfies the identity

$$\int_0^T \int_0^\infty \{(P_x)^m \varphi_x - P\varphi_t\}\, dxdt = \int_0^\infty P_0(x)\varphi(x,0)\, dx + \int_0^T A^m \varphi(0,t)\, dt,$$

for all $\varphi \in C^1(\bar{H}_T)$, which vanish for $x = 0$ and for large values of x and $t = T$.

The same methods as used by Oleinik *et al.* (1958), Aronson (1969), Gilding and Peletier (1976) and Nagia and Mimura (1983) are used here to obtain the solutions of Problem I and Problem II as the limit of a sequence of classical solutions to an approximate non-degenerate equation with the initial-boundary value conditions. A crucial step is to derive a Hölder estimate for the approximate solutions. In section 2, the uniqueness of solutions of Problem I and Problem II is discussed. In section 3 and section 4, some estimations of the approximate solutions for Problem I and Problem II are established. These estimations enable us to prove the existence of solutions for the two problems in section 5. Finally, in section 6, the regularity of solutions of these two problems will be stated.

2. Uniqueness.

Theorem 2.1. *Problem I has at most one solution satisfying Definition 1.1.*

Proof. We first remark that the solution $P(x,t)$ satisfying Definition 1.1 for Problem I also satisfies

$$((P_x)^m)_x, \quad P_t \in L^\infty_{loc}(R^+ \times [\tau, T]) \ \text{ for any } \ 0 < \tau < T < \infty,$$

and

$$P_t = ((P_x)^m)_x \ \text{ a.e. in } \ R^+ \times (0, \infty). \tag{6-2-1}$$

Let $P(x,t)$ and $q(x,t)$ be solutions of Problem I with the same initial and boundary value conditions. Considering the remark stated above, we have

$$(P-q)_t = [(P_x)^m - (q_x)^m]_x \ \text{ a.e. in } \ R^+ \times (0, \infty). \tag{6-2-2}$$

We define a function $\chi_N(x) \in C^\infty(\bar{R}^+)$ such that

$$
\begin{aligned}
&0 \le \chi_N(x) \le 1, \\
&\chi_N(x) = 1 \ \text{ for } \ 0 \le x \le N, \ \ \chi_N(x) = 0 \ \text{ for } \ x > N+1, \\
&|\chi_N'(x)| \le 2 \ \text{ for all } \ N \ \text{ and all } \ x \in [0,\infty).
\end{aligned}
$$

We multiply (6-2-2) by $[P(x,t)-q(x,t)]\chi_N(x)$ and integrate over $R^+ \times [\tau, T]$, where positive τ and T are arbitrarily fixed. Integrating by parts the resulting equation yields

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty [P(x,T)-q(x,T)]^2 \chi_N(x)\,dx \\
&\quad + \int_\tau^T \int_0^\infty [(P_x)^m - (q_x)^m](P-q)_x \chi_N(x)\,dxdt \\
&= \frac{1}{2}\int_0^\infty [P(x,\tau)-q(x,\tau)]^2 \chi_N(x)\,dx \\
&\quad - \int_\tau^T \int_0^\infty [(P_x)^m - (q_x)^m](P-q)\chi_N'(x)\,dxdt.
\end{aligned} \tag{6-2-3}
$$

Since $P(x,t)$ and $q(x,t)$ are the solutions satisfying Definition 1.1, we set $0 < P,\ q \le k_1,\ 0 \le P_x,\ q_x \le k_2$ and write

$$
A(x,t) = \int_0^1 m(\theta P_x + (1-\theta)q_x)^{m-1}\,d\theta.
$$

Obviously $0 \le A(x,t) \le mk_1^{m-1}$. Letting $\tau \to 0$ in (6-2-3) and noting $P(x,0) = q(x,0)$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty [P(x,T)-q(x,T)]^2 \chi_N(x)\,dx \\
&\quad + \int_0^T \int_0^\infty A(x,t)[(P-q)_x]^2 \chi_N(x)\,dxdt \\
&= -\int_0^T \int_0^\infty A(x,t)(P-q)_x \chi_N'(x)\,dxdt.
\end{aligned} \tag{6-2-4}
$$

We can estimate the right hand side of (6-2-4) by

$$
\begin{aligned}
&-\int_0^T\int_0^\infty A(x,t)(P-q)_x(P-q)\chi_N'(x)\,dxdt\\
&\le \int_0^T\int_{N\le x\le N+1}|A(x,t)(P-q)_x(P-q)\chi_N'(x)|\,dxdt\\
&\le 2\int_0^T\{[\int_{N\le x\le N+1}A(x,t)(P-q)_x^2\,dx][\int_{N\le x\le N+1}A(x,t)(P-q)^2\,dx]\}^{\frac{1}{2}}\,dt=I.
\end{aligned}
\tag{6-2-5}
$$

First, we note that the term on the right side of (6-2-5) denoted by I has a bound

$$I \le 8Tmk_2^m k_1. \tag{6-2-6}$$

Consequently, from (6-2-6), (6-2-5) and (6-2-4) we conclude by use of the monotone convergence theorem that

$$\int_0^\infty [P(x,T)-q(x,T)]^2\,dx<\infty,$$

and

$$\int_0^T\int_0^\infty A(x,t)[(P-q)_x]\,dxdt<\infty.$$

Therefore, an alternative estimation of I could be given by

$$I\le\int_0^T\int_0^\infty A(x,t)[(P-q)_x]^2\,dxdt+mk_2^{m-1}\int_0^T\int_0^\infty(P-q)^2\,dxdt. \tag{6-2-7}$$

Substituting (6-2-5) with (6-2-7) into (6-2-4) and letting $N\to\infty$, we have

$$\int_0^\infty [P(x,T)-q(x,T)]^2\,dx\le 2mk_2^{m-1}\int_0^T\int_0^\infty[P(x,t)-q(x,t)]^2\,dxdt.$$

This implies, using Gronwall's lemma (Ladyzenskaja *et al.*, p. 94), that

$$[P(x,T)-q(x,T)]^2=0 \ \text{ for } \ x\in(0,\infty),$$

for any $T\in(0,\infty)$. Thus the proof is completed.

By the same techniques as used in the proof of Theorem 2.1, we can also prove a uniqueness result for the solution of Problem II. We omit the proof and state the following theorem.

Theorem 2.2. *Problem II has at most one solution satisfying Definition 1.2.*

3. Some Auxiliary Results for Problem I.

We suppose that $P_0(x)$ satisfies the following hypotheses:

(A_2) (i) $P_0(x)$ is an infinitely differentiable function defined on [0,n];

(ii) $P_0'(x) \geq 0$ for $x \in [0, n]$;

(iii) $P_0(x) = P_e$ for $n - 1 \leq x \leq n$;

(iv) $P_0(x) = P_w$ for $0 \leq x \leq 1/n$.

For a positive integer n and a positive number T, we define

$$H_n = (0, n) \times (0, \infty) \text{ and } H_n^T = (0, n) \times (0, T).$$

We consider the following problem:

$$P_t = ((P_x + \epsilon)^m)_x \qquad \text{in } H_n, \tag{6-3-1}$$

$$\text{I'} \qquad P(0,t) = P_w, \;\; P(n,t) = P_e, \;\; 0 \leq t \leq \infty, \tag{6-3-2}$$

$$P(x,0) = P_0(x) \qquad 0 \leq x \leq n, \tag{6-3-3}$$

for any fixed and sufficiently small ϵ. The solution $P_n^\epsilon(x,t)$ of Problem I' depends on n and ϵ. For simplicity, we still write $P(x,t)$ instead of $P_n^\epsilon(x,t)$ in this section.

Lemma 3.1. *Problem I' has a unique classical solution $P(x,t)$ satisfying the following properties:*

(i) $P_w \leq P \leq P_e$ in $\bar{H}_n$;

(ii) $P_x \geq 0$ in $\bar{H}_n$ and $P_x > 0$ in in H_n;

(iii) there exists an α with $0 < \alpha \leq 1$ such that $P \in C^{2+\alpha, 1+\alpha/2}(\bar{H}_n)$ for any T with $0 < T < \infty$;

(iv) $P_{xx} \in C^{2,1}(H_n)$.

Proof. Let $f(s)$ be the smooth function on R^1 such that $f(s) = m(S+\epsilon)^{m-1}$ for $S \geq 0$, $f(s) \geq m(\epsilon/2)^{m-1}$ on R^1 and there are the positive constants ν and μ satisfying

$$\nu(|s|+\epsilon)^{m-1} \leq f(s) \leq \mu(|s|+\epsilon)^{m-1} \quad \text{on } R^1.$$

Then for the equation

$$P_t = f(P_x)P_{xx} \quad in \;\; H_n, \tag{6-3-4}$$

and the initial-boundary conditions (6-3-2) and (6-3-3), Theorem 4.1 of Ladyzenskaja *et al.* (1968, p. 558) shows that there exists a unique function $P(x,t)$ which has property (iii).

Standard maximum principle implies that property (i) is true. Property (iv) can be shown by virtue of a regularity argument as stated in the book of Friedman (1964).

To verify property (ii), we differentiate equation (6-3-4) with respect to x and write $u = P_x$. Then u satisfies the equation

$$u_t = f(P_x)u_{xx} + (f'(P_x)P_{xx})u_x. \tag{6-3-5}$$

Using property (i) and the boundary conditions (6-3-2), we see that

$$u(0,t) \geq 0 \;\text{ and }\; u(n,t) \geq 0 \quad \text{for all } t \in (0,\infty),$$

and

$$u(x,0) = P_0'(x) \geq 0 \quad \text{for } x \in [0,n].$$

A second application of maximum principle now yields

$$u(0,t) \geq 0 \;\text{ in }\; \bar{H}_n \;\text{ and }\; u(x,t) > 0 \;\text{ in }\; H_n,$$

which is property (ii). This completes the proof.

Next we shall give the bounds of P_x which are independent of n and ϵ.

Lemma 3.2. *Let $P(x,t)$ be a solution of Problem I, then we have*

$$0 \le P_x \le C_1 \ \ in \ \ \bar{H}_n,$$

where C_1 is a constant depending only on P_w, P_e and $\|P_0'(x)\|_{L^\infty}$.

Proof. Because of property (ii) of Lemma 3.1, what we only need to show is that P_x is bounded from above. For this purpose, we shall construct the some suitable comparison functions.

First, we conclude that P_x is bounded from above on the lateral boundaries $x = 0$ and $x = n$. Let us assume that

$$0 \le P_0'(x) \le L_1 \ \text{ for } x \in [o, n], \tag{6-3-6}$$

we can choose a constant L such that $L \ge L_1$ and $L + P_w \ge P_e$, and define a linear function as

$$q_1(x,t) = Lx + P_w.$$

It is obvious that $q_1(x,t)$ satisfies equation (6-3-1). By writing

$$z_1(x,t) = q_1(x,t) - P(x,t),$$

then $z_1(x,t)$ satisfies the equation

$$z_{1t} = m[(P_x + \epsilon)^{m-1}]z_{1xx} \ \text{ in } \ H_n.$$

From the definition of $z_1(x,t)$, it is easy to observe that on $t = 0$

$$z_1(x,0) = Lx + P_w - P_0(x) \ge 0,$$

and on the lateral boundary $x = 0$

$$z_1(n,t) = Ln + P_w - P_e \ge 0.$$

Applying the maximum principle, we obtain

$$z_1(x,t) \geq 0 \;\text{ in }\; \bar{H}_n.$$

This implies that

$$P(x,t) \leq Lx + P_w \;\text{ in }\; \bar{H}_n.$$

Thus, we have

$$P_x(0,t) = \lim_{x\to 0+} \frac{P(x,t) - P_w}{x} \leq L.$$

Next we define a function by

$$q_2(x,t) = P_e(x - n + 1).$$

Obviously, $q_2(x,t)$ also satisfies equation (6-3-1) in H_n.

On setting

$$z_2(x,t) = P(x,t) - q_2(x,t),$$

then $z_2(x,t)$ satisfies the equation

$$z_{2t} = m[(P_x + \epsilon)^{m-1}]z_{2xx} \;\text{ in }\; H_n.$$

Now we apply the maximum principle on $[n-1,n]\times[0,T]$ for any T with $0 < T < \infty$. From property (i) of Lemma 3.1 and the initial-boundary conditions, it follows that

$$z_2(n-1,t) = P(n-1,t) \geq P_w > 0 \;\text{ for }\; 0 \leq t \leq T,$$

$$z_2(n,t) = 0 \;\text{ for }\; 0 \leq t \leq T,$$

$$z_2(x,t) = P_e(n-x) \;\text{ for }\; n-1 \leq x \leq n.$$

An application of the maximum principle yields $z_2(x,t) \geq 0$ on $[n-1,n]\times[0,T]$, which also implies that $z_2(x,t)$ attains its minimum on the lateral boundary $x = n$. Hence, we have

$$z_{2x}(n,t) \leq 0,$$

and so

$$P_x(n,t) \le P_e.$$

Finally, we consider equation (6-3-5). By using (6-3-6) to (6-3-8), and as a consequence of the maximum principle, we derive

$$P_x \le C_1 = \max(L, P_e),$$

which is the desired conclusion.

Lemma 3.3. *For any $\tau \in (0,\infty)$ and any fixed small $\delta > 0$, there holds*

$$|((P_x + \epsilon)^m)_x| \le C_2 \quad \text{on} \quad [\delta, n-1] \times (\tau, \infty), \tag{6-3-9}$$

where C_2 is a constant depending only on $m, P_w, P_e, \|P_0'(x)\|_{L^\infty}, \delta$ and τ.

Proof. We use a Bernstein-type technique to get this estimation. Differentiating (6-3-1) with respect to x and putting $u = P_x$, we have

$$u_t = [m(u+\epsilon)^{m-1}u_x]_x \quad \text{in } H_n. \tag{6-3-10}$$

It follows from (ii) of Lemma 3.1 that $u \ge 0$ in $\bar{H}_n$ and $u > 0$ in H_n.

Define the function $\phi(u)$ by

$$\phi(u) = \int_0^u \frac{a(s)}{\theta(s)}\, ds \quad \text{for} \quad 0 \le u \le C_1,$$

where C_1 is the constant used in Lemma 3.2. The form of $\phi(u)$ is the one introduced by Gilding (1976). Here a and θ are, respectively, specified as

$$a(s) = m(s+\epsilon)^{m-1}$$

and

$$\theta(s) = [\int_0^s r a'(r)\, dr + 2sa(C_1) - sa(s) + s + 1]^{1/2},$$

for $0 \le s \le C_1$. It is easy to verify the following relations:

$$
\begin{aligned}
&\theta'(s) = \frac{1}{2}[2a(C_1) - a(s) + 1]\frac{1}{\theta(s)} > 0,\\
&\theta''(s) = -\frac{1}{2}[a'(s) + 2(\theta'(s))^2]\frac{1}{\theta(s)} < 0,\\
&0 \le a'(s)\theta(s) \le -2\theta^2(s)\theta''(s), \qquad\qquad (6\text{-}3\text{-}11)\\
&0 \le a(s)\theta'(s) \le -\theta^2(s)\theta''(s),\\
&0 \le \frac{a(s)}{\theta(s)\theta''} \le \frac{2}{m-1}(s+\epsilon).
\end{aligned}
$$

We define the function $w(x,t)$ by

$$w(x,t) = \phi(u(x,t)) \quad \text{for} \quad (x,t) \in \bar{H}_n. \qquad (6\text{-}3\text{-}12)$$

Substituting (6-3-12) into (6-3-10), we obtain

$$w_t = a(u)w_{xx} + \theta'(u)(w_x)^2.$$

Differentiating this equation with respect to x, multiplying it by w_x and writing $q = w_x$, we get

$$
\begin{aligned}
&\frac{1}{2}(q^2)_t - a(u)qq_{xx}\\
&= [\frac{a'(u)}{a(u)}\theta(u) + 2\theta'(u)]q^2q_x + \frac{1}{a(u)}\theta(u)\theta''(u)q^4. \qquad (6\text{-}3\text{-}13)
\end{aligned}
$$

Now we consider the function $Z(x,t) = \chi^2(x,t)q^2(x,t)$ in which χ is a smooth cut-off function on $[0,\infty) \times [0,\infty)$ such that $0 \le \chi(x,t) \le 1$ on $[0,\infty) \times [0,\infty)$, $\chi(x,t) = 1$ on $[\delta, n-1] \times [\tau, \infty)$ and $\chi(x,t) = 0$ on the outside of $[\frac{1}{2}\delta, n - \frac{1}{2}] \times [\frac{1}{2}\tau, \infty)$, where τ is any fixed constant with $0 < \tau < \infty$. For an arbitrary fixed T with $0 < \tau < T < \infty$, if Z attains a positive maximum at point $(x_0, t_0) \in H_n^T$, then at this point we have

$$Z_x = 0 \quad \text{and} \quad a(u)Z_{xx} - Z_t \le 0,$$

or, in other words

$$\chi q_x = -\chi_x q \tag{6-3-14}$$

and

$$\chi^2\{\frac{1}{2}(q^2)_t - a(u)qq_{xx}\}$$
$$\geq a(u)\{\chi^2(q_x)^2 + 4\chi\chi_x qq_x + \chi q_{xx}q^2 + (\chi_x)^2q^2\} - \chi\chi_t q^2. \tag{6-3-15}$$

Substituting (6-3-13) into (6-3-15) and using (6-3-14), we have

$$-\frac{1}{a(u)}\theta(u)\theta''(u)\chi^2q^4$$
$$\leq -[\frac{a'(u)}{a(u)}\theta(u) + 2\theta']\chi_x\chi q^3 + \{a(u)[2(\chi_x)^2 - \chi\chi_{xx}] + \chi\chi_t\}q^2$$

Noting $\theta'' < 0$, we also get

$$\chi^2q^4 \leq [\frac{a'(u)}{\theta''(u)} + \frac{2a(u)\theta'(u)}{\theta(u)\theta''(u)}]\chi_x\chi q^3$$
$$-\frac{a(u)}{\theta(u)\theta''(u)}\{a(u)[2(\chi_x)^2 - \chi\chi_{xx}] + \chi\chi_t\}q^2.$$

It is sufficient to assume $|q| \geq 1$ here. It follows from (6-3-11) that

$$|\frac{a'(u)}{\theta''(u)} + \frac{2a(u)\theta'(u)}{\theta(u)\theta''(u)}| \leq 4\theta(C_1),$$
$$|\frac{a^2(u)}{\theta(u)\theta''(u)}| \leq \frac{2}{m-1}(C_1+1)a(C_1). \tag{6-3-16}$$

Combining (6-3-16) with the inequalities just gained and noting $|q| \geq 1$, we see that

$$(\chi q)^2 \leq M_1,$$

where M_1 is a positive constant depending only on C_1, δ and τ. We also note that

$$((u+\epsilon)^m)_x = a(u)u_x = \theta(u)w_x.$$

Hence, at the point (x_0, t_0) we obtain

$$|((u+\epsilon)^m)_x| \leq \theta(C_1)M_1^{\frac{1}{2}} \leq \{[2mC_1(C_1+1)^{m-1} + C_1 + 1]M_1\}^{\frac{1}{2}}.$$

Putting $C_2 = \{[2mC_1(C_1)^{m-1} + C_1 + 1]M_1\}^{\frac{1}{2}}$, we complete the proof.

To show the regularity results we need the following lemma.

Lemma 3.4. *For any $\tau \in (0,\infty)$ and any fixed small $\delta > 0$, there holds*

$$|((P_x+\epsilon)^{m-1})_x| \geq C_2 \quad on \quad [\delta, n-1] \times (\tau, \infty), \tag{6-3-17}$$

where C_2 is a constant depending only on $m, P_w, P_e, \|P_0^{\prime(x)}\|_{L^\infty}, \delta$ and τ.

Proof. Differentiating equation (6-3-1) with respect to x and putting $u = P_x$, we then have

$$u_t = ((u+\epsilon)^m)_{xx}. \tag{6-3-18}$$

By denoting $w = (u+\epsilon)^{m-1}$, this equation is rewritten as

$$w_t = mww_{xx} + \frac{m}{m-1}(w_x)^2.$$

For $0 \leq r \leq 1$, letting $\phi(r) = \frac{Nr}{3}(4-r)$, $N = (C_1+1)^{m-1}$, we define the function $z(x,t)$ by

$$w(x,t) = \phi(z(x,t)) \quad \text{on} \quad [0,n] \times [\tau, \infty).$$

It follows from (6-3-18) that

$$z_t - m\phi z_{xx} = m[\phi\frac{\phi''}{\phi'} + \frac{1}{m-1}\phi'](z_x)^2.$$

By using almost the same technique used to prove Lemma 3.3 we can derive the assertion of this lemma, so we omit the details of the proof here.

In order to show that the Hölder continuity of $P(x,t)$ with respect to t holds independently of n and ϵ, we use the following results due to Gilding (1976).

Lemma 3.5. *Let $z(x,t) \in C^{2,1}((a,b)\times(\tau,T)) \cap C^0([a,b]\times[\tau,T])$ be a solution of the equation*

$$z_t = A(x,t)z_{xx} + B(x,t)z_x + f(x,t) \quad \text{in} \quad (a,b)\times(\tau,T)$$

where $-\infty < a < b < \infty$, $0 \leq \tau < T < \infty$, and let A, B and f be continuous on $[a,b] \times [\tau, T]$ such that

$$0 < A(x,t) \leq \mu, \ |B(x,t)| \leq \mu \text{ and } |f(x,t)| \leq \mu \ \ in \ \ [a,b] \times [\tau, T]$$

for some positive constant μ. If z is Hölder continuous with respect to x in $[a,b] \times [\tau, T]$ with an exponent $\alpha \in (0,1]$ and a Hölder constant M_1, then for any $0 < d < (b-a)/2$ it holds that

$$|z(x,s) - z(x,t)| \leq M_2 |s-t|^{\alpha/2},$$

for $\tau \leq s < t \leq s + \delta \leq T$ and $x \in [a+d, b-d]$, where $\delta = \frac{d^2}{4\mu(1+d)}$ and $M_2 = 2\{M_1[2\mu(1+d)^{1/2}]^{\alpha} + \mu\delta^{1-\alpha/2}\}$.

We also need an elementary result which was used by van Duyn and Peletier (1976).

Lemma 3.6. *Let $f \in C^1([0,1])$ have the following properties: (i) $|f'| < A$ on [0,1], (ii)* $|\int_a^b f(x)\,dx| \leq B$ for any $a, b \in [0,1]$. *Then* $|f(x)| \leq \max\{2B, \sqrt{2AB}\}$ for $0 \leq x \leq 1$.

Combining Lemma 3.5 and Lemma 3.6, we arrive at

Lemma 3.7. *Let $P_0(x)$ satisfy the hypotheses (A_2), $P(x,t)$ be a solution of Problem I', then $P(x,t)$ satisfies the following condition:*

$$|P(x,s) - P(x,t)| \leq C_4 |s-t|^{1/2} \ \ \text{on } [0, n-1] \times [0, \infty), \tag{6-3-19}$$

where C_4 is a constant depending only on P_w, P_e, m and $\|P_0'(x)\|_{L^\infty}$.

Proof. Equation (6-3-1) is rewritten as

$$P_t = m(P_x + \epsilon)^{m-1} P_{xx} \ \text{ in } (0,n) \times (\tau, T),$$

for any $0 \le \tau < T < \infty$. It follow from Lemma 3.2 that

$$0 < m(P_x + \epsilon)^{m-1} \le m(C_1 + 1)^{m-1} \quad \text{on} \quad [0, n] \times [\tau, T],$$

and

$$|P(x,t) - P(y,t)| \le C_1|x - y| \quad \text{on} \quad [0, n] \times [\tau, T].$$

Thus, Lemma 3.5 yields

$$|P(x,s) - P(x,t)| \le C_4|s - t| \quad \text{on} \quad [d, n-1] \times [\tau, T], \tag{6-3-20}$$

for a small positive number d, where C_4 is the constant depending on d, m, P_e, P_w and $\|P_0'(x)\|_{L^\infty}$. We can assume that $0 < d < 1$ for a fixed d. In order to get the result of this lemma, we only need to prove that $P(x,t)$ is Hölder continuous with respect to t on $x \in [0,1]$.

For any $a, b \in [0,1]$, $t_1, t_2 \in [\tau, T]$, we define the rectangle $R = (a,b) \times (t_1, t_2)$ and integrate equation (6-3-1) over R. This yields

$$\int_a^b \{P(x,t_2) - P(x,t_1)\}\,dx = \int_{t_1}^{t_2} \{(P_x + \epsilon)^m(b,t) - (P_x + \epsilon)^m(a,t)\}\,dt.$$

Hence, by Lemma 3.2 we conclude that

$$|\int_a^b \{P(x,t_2) - P(x,t_1)\}\,dx| \le 2(C_1 + 1)^m |t_2 - t_1|^{1/2}.$$

We now apply Lemma 3.6 with $f(x) = P(x,t_2) - P(x,t_1)$. Since $|f(x)| \le 2C_1$, when $2C_1 \ge 4(C_1+1)^m|t_2 - t_1|$, it follows from Lemma 3.6 that

$$|P(x,t_2) - P(x,t_1)| \le (8C_1)^{1/2}(C_1 + 1)^{m/2}|t_2 - t_1|^{1/2}. \tag{6-3-21}$$

On the other hand, when $2C_1 < 4(C_1 + 1)^m|t_2 - t_1|$, we have

$$|P(x,t_2) - P(x,t_1)| \le (P_e - P_w) \le (P_e - P_w)[\frac{2(C_1+1)^m}{C_1}]^{1/2}|t_2 - t_1|^{1/2}. \tag{6-3-22}$$

Thus, from (6-3-20) to (6-3-22) we note that the conclusion of this lemma is true.

Lemma 3.8. *For any $\tau \in (0,\infty)$ and any fixed small $\delta > 0$, $P(x,t)$ satisfies the following conditions*

$$(i) \quad |(P_x+\epsilon)^m(x,s) - (P_x+\epsilon)^m(x,t)| \le C_5|s-t|^{1/2} \quad \text{on } [2\delta, n-2] \times [\tau,\infty),$$

where C_5 is a constant depending only on $m, P_w, P_e, \|P_0'(x)\|_{L^\infty}, \delta$ and τ.

$$(ii) \quad |(P_x+\epsilon)^{m-1}(x,s) - (P_x+\epsilon)^{m-1}(x,t)| \le C_6|s-t|^{1/2} \quad \text{on } [2\delta, n-2] \times [\tau,\infty),$$

where C_6 is a constant depending only on $m, P_w, P_e, \|P_0'(x)\|_{L^\infty}, \delta$ and τ.

This lemma can be directly derived as corollaries of Lemma 3.3, Lemma 3.4 and Lemma 3.5. Summarizing the above results in this section, we have established the following theorem.

Theorem 3.1. *Let $P_0(x)$ satisfy the hypotheses (A_2). Problem I' has a unique classical solution $P(x,t)$ such that*

(i) $P_w \le P \le P_e$ in $\bar{H}_n$.

(ii) There is a constant $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty})$ such that $0 \le P_x \le C$ in $\bar{H}_n$.

(iii) There exists an $\alpha (0 < \alpha \le 1)$ such that $P \in C^{2+\alpha, 1+\alpha/2}(\bar{H}_n^T)$ for any $T \in (0,\infty)$.

(iv) $P_{xx} \in C^{2,1}(H_n)$.

(v) There is a constant $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty})$ such that

$$|P(x,s) - P(y,t)| \le C[|x-y| + |s-t|^{1/2}],$$

for any $0 \le x,\ y \le n$ and $0 \le s,\ t < \infty$.

(vi) For any $\tau \in (0,\infty)$, any small $\eta > 0$, there is a constant $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty}, \eta, \tau)$ such that

$$|(P_x+\epsilon)^m(x,s) - (P_x+\epsilon)^m(y,t)| \le C[|x-y| + |s-t|^{1/2}],$$

for $\eta \leq x,\ y \leq n-2,\ \tau \leq s,\ t < \infty$.

(vii) For any $\tau \in (0,\infty)$, *any small* $\eta > 0$, *there is a constant* $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty}, \eta, \tau)$ *such that*

$$|(P_x+\epsilon)^{m-1}(x,s) - (P_x+\epsilon)^{m-1}(y,t)| \leq C[|x-y| + |s-t|^{1/2}],$$

for $\eta \leq x,\ y \leq n-2,\ \tau \leq s,\ t < \infty$.

4. Some Auxiliary Results for Problem II.

In this section we assume that $P_0(x)$ satisfies the following hypotheses:

(A_3) (i) $P_0(x)$ is an infinitely differentiable function on $[0,n]$;

(ii) $P_0'(x) \geq 0$ for $x \in [0,n]$;

(iii) $P_0(x) = P_e$ for $n-1 \leq x \leq n$;

(iv) $P_0'(x) = A$ for $0 \leq x \leq \frac{1}{n}$ and $P_0(0) > 0$.

Now we consider following initial-boundary value problem:

$$P_t = ((P_x+\epsilon)^m)_x \qquad \text{in } H_n^T, \tag{6-4-1}$$

$$\text{II'} \qquad P_x(0,t) = A,\ \ P_x(n,t) = 0,\ \ 0 \leq t \leq T, \tag{6-4-2}$$

$$P(x,0) = P_0(x) \qquad 0 \leq x \leq n, \tag{6-4-3}$$

As we did in section 3, we still write $P(x,t)$ to denote the solution $P_n^\epsilon(x,t)$ of Problem II'. We can establish the following theorem:

Theorem 4.1. *Let* $P_0(x)$ *satisfy the hypotheses* (A_3). *There exists a* $T^* > 0$, *for any* $T \leq T^*$, *Problem II' has a unique classical solution* $P(x,t)$ *such that*

(i) $0 \leq P(x,t) \leq P_e$ *in* $\bar{H}_n^T$ *in* $\bar{H}_n$.

(ii) There is a constant $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty})$ *such that* $0 \leq P_x \leq C$ *in* $\bar{H}_n$.

(iii) There exists an $\alpha(0 < \alpha \leq 1)$ *such that* $P \in C^{2+\alpha,1+\alpha/2}(\bar{H}_n^T)$ *for any* $T \in (0, \infty)$.

(iv) $P_{xx} \in C^{2,1}(H_n)$.

(v) There is a constant $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty})$ *such that*

$$|P(x,s) - P(y,t)| \leq [|x-y| + |s-t|^{1/2}],$$

for $0 \leq x,\ y \leq n,\ 0 \leq s,\ t < T$.

(vi) For any $\tau \in (0,T)$, *any small* $\eta > 0$, *there is a constant* $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty}, \eta, \tau)$ *such that*

$$|(P_x + \epsilon)^m(x,s) - (P_x + \epsilon)^m(y,t)| \leq C[|x-y| + |s-t|^{1/2}],$$

for $\eta \leq x,\ y \leq n,\ \tau \leq s,\ t \leq T$.

(vii) For any $\tau \in (0,T)$, *any small* $\eta > 0$, *there is a constant* $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty}, \eta, \tau)$ *such that*

$$|(P_x + \epsilon)^{m-1}(x,s) - (P_x + \epsilon)^{m-1}(y,t)| \leq C[|x-y| + |s-t|^{1/2}],$$

for $\eta \leq x,\ y \leq n,\ \tau \leq s,\ t \leq T$.

Proof. To prove this theorem we use arguments which are analogous to the ones used in section 3. We therefore only sketch the proof and only give the detail where there are differences from those in Problem I'.

First, for any $T \in (0, \infty)$ we consider Problem II' in $\bar{H}_n^T$. According to the results in Chapter V of Ladyzenskaja *et al.* (1968), we know that there exists a unique solution of Problem II' satisfying property (iii). As $P_0(x)$ satisfies the hypotheses (A_3) it follows from the linear theory of parabolic equations that property (iv) holds.

Differentiating equation (6-4-1) with respect to x and using the initial and boundary conditions, we have the property (ii) by maximum principle.

From physical considerations of our problem, we are interested in non-negative solutions of Problem II'. Because $P_0(0) > 0$ and $P(x,t)$ is a continuous function in $\bar{H}_n^T$, there exists a $T^* > 0$ such that $P(0,T^*) = 0$ and $P(0,t) > 0$ for $0 < t \leq T^*$. We now discuss the bound on solution $P(x,t)$ of Problem II' in $\bar{H}_n^T$ for $0 < T \leq T^*$. From property (ii) proved above, we see that $P_x \geq 0$ in $\bar{H}_n^T$ and $P(0,t) > 0$ for $t < T^*$, and it is obvious that $P \geq 0$ in $\bar{H}_n^T$ for $T \geq T^*$. In order to prove that the solution $P(x,t)$ has a bound from above, which is independent of n and ϵ, we construct a function defined as

$$v = (P - P_e - \frac{\delta}{n}x)e^{-\lambda t}, \quad \lambda > o,$$

where δ is an arbitrary sufficiently small positive number. Then v satisfies the equation

$$v_t - m(P_x + \epsilon)^{m-1} + \lambda v = 0. \tag{6-4-4}$$

We want to prove that the function v cannot attain a positive maximum value at any point in $\bar{H}_n^T$. Since $v \leq 0$ on $t = 0$, $v_x = (P_x - \frac{\delta}{n})e^{-\lambda t} = (A - \frac{\delta}{n})e^{-\lambda t} > 0$ on the lateral boundary $x = 0$ and $v_x = (P_x - \frac{\delta}{n})e^{-\lambda t} < 0$ on $x = n$, it is not possible that v has its positive greatest value at a point either on the bottom in $\bar{H}_n^T$, or on the lateral boundary in $\bar{H}_n^T$. On the other hand, if v attains its positive greatest value at a point on the interior of H_n^T , or on the top $t = T$, at this point we have

$$v_t > 0 \quad \text{and} \quad m(P_x + \epsilon)^{m-1}v_{xx} \leq 0.$$

Thus from (6-4-4) it follows that

$$-\lambda v = v_t - m(P_x + \epsilon)^{m-1}v_{xx} \geq 0.$$

This implies $v \leq 0$, which yields a contradiction. So we have proven that v cannot attain a positive maximum in $\bar{H}_n^T$, hence $v \leq 0$, i.e.,

$$P \leq P_e + \frac{\delta}{n} x \leq P_e + \delta.$$

Since δ is arbitrarily small, this implies that $P \leq P_e$. We now get property (i).

By the same method as used in section 3, we can prove that all the properties from (v) to (vii) are true.

5. Existence.

Based on the auxiliary results in the above two sections, we are now in a position to prove the existence theorems for Problem I and Problem II.

Theorem 5.1. *Problem I has a unique solution satisfying Definition 1.1, moreover, this solution has the following properties:*

(i) There is a constant $C = C(P_w, P_e, \|P_0'(x)\|_{L^\infty})$ such that

$$|P(x,s) - P(y,t)| \leq [|x-y| + |s-t|^{1/2}].$$

for $0 \leq x,\ y < \infty,\ 0 \leq s,\ t < \infty$.

(ii) For any $\tau \in (0,T)$, any small $\eta > 0$, there is a constant $C = C(m, P_w, P_e, \|P_0'(x)\|_{L^\infty}, \eta, \tau)$ such that

$$|(P_x+\epsilon)^m(x,s) - (P_x+\epsilon)^m(y,t)| \leq C[|x-y| + |s-t|^{1/2}],$$

for $\eta \leq x,\ y < \infty,\ \tau \leq s,\ t < \infty$.

Theorem 5.2. *Problem II has a unique solution satisfying Definition 1.2, moreover, this solution has the following properties:*

(i) There is a constant $C = C(A, P_e, \|P_0'(x)\|_{L^\infty})$ such that

$$|P(x,s) - P(y,t)| \leq C[|x-y| + |s-t|^{1/2}].$$

for $0 \le x,\ y < \infty,\ 0 \le s,\ t < \infty.$

(ii) For any $\tau \in (0,T)$, *any small* $\eta > 0$, *there is a constant* $C = C(m, A, P_e, \|P_0'(x)\|_{L^\infty}, \eta, \tau)$ *such that*

$$|(P_x + \epsilon)^m(x,s) - (P_x + \epsilon)^m(y,t)| \le C[|x-y| + |s-t|^{1/2}],$$

for $\eta \le x,\ y < \infty,\ \tau \le s,\ t < \infty.$

Proof of Theorem 5.1 We suppose that $P_0(x)$ satisfies the hypotheses (A_1) and $P_0(0) = P_w$. Let $\tilde{P}_0(x)$ be a continuous extension of $P_0(x)$ such that $\tilde{P}_0(x) = P_0(x)$ for $x \ge 0$ and $\tilde{P}_0(x) = P_w$ for $x < 0$. Let $\alpha(x) \in C_0^\infty(R^1)$ be a non-negative function such that $\alpha(x) = 0$ for $|x| \ge 1$ and $\int_{R^1} \alpha(x)\,dx = 1$. Then for each $n \ge 1$, we set $\alpha_{1/n}(x) = n\alpha(nx)$ and define

$$P_{n0}(x) = \int_{R^1} \alpha_{1/n}(x-y)\tilde{P}_0(y - \frac{2}{n})\,dy.$$

It is easy to verify that $P_{n0}(x)$ satisfies the hypotheses (A_2) if $n > 2$ and $P_{n0}(x) \to P_0(x)$ uniformly on $\bar{R}^+$ as $n \to \infty$. In addition, $\|P_{n0}'(x)\|_{L^\infty} \le \|P_0'(x)\|_{L^\infty}$. Using $P_{n0}(x)$ as an initial function, we consider Problem I'. It follows from Theorem 3.1 that Problem I' has a unique solution $P_n^\epsilon(x,t)$ satisfying properties (i)-(vii) of this theorem. For large N and an arbitrary $T > 0$, when $n > N$, applying Ascoli-Arzela's theorem and a diagonal process, from $\{P_n^\epsilon(x,t)\}$ we can select a subsequence $\{P_{n_j}^{\epsilon_j}(x,t)\}$ which converges to a limit function $P(x,t)$ uniformly on $[0,N] \times [0,T]$ as $\epsilon_j \to 0,\ n_j \to \infty$.

Moreover, from the properties (ii) and (vi) of Theorem 3.1 we observe that

$$P_{n_j x}^{\epsilon_j}(x,t) \to P_x(x,t) \text{ uniformly on any compact subset of } (0,N) \times (0,T)$$

as $\epsilon_j \to 0,\ n_j \to \infty$ and

$$(P_{n_j x}^{\epsilon_j}(x,t) + \epsilon_j)^m \to (P_x(x,t))^m \text{ weakly in } L^2((0,N) \times (0,T)),$$

as $\epsilon_j \to 0$, $n_j \to \infty$. We multiply the equation (6-3-1), where $P(x,t)$ should be rewritten as $P_{n_j}^{\epsilon_j}(x,t)$, by any $\varphi \in C^1(\bar{H})$, which vanish for $x = 0$ and for large values of x and t. Using integration by parts and letting $\epsilon_j \to 0$, $n_j \to \infty$, we obtain that the limit function $P(x,t)$ satisfying the integral identity

$$\int_0^\infty \int_0^\infty \{(P_x)^m \phi_x - P\phi_t\}\, dxdt = \int_0^\infty P_0(x)\phi(x,0)\, dx.$$

The limit function $P(x,t)$ obviously satisfies all the requirements of Definition 1.1 and the properties of Theorem 5.1.

Proof of Theorem 5.2 We suppose that $P_0(x)$ satisfies the hypotheses (A_1) and $P_0(0) > 0$, $P_0'(0) = A$. Let $\tilde{P}_0(x)$ be a continuous extension of $P_0(x)$ such that $\tilde{P}_0(x) = P_0(x)$ for $x \geq 0$ and $\tilde{P}_0(x) = P_0(0) + Ax$ for $x < 0$. We define

$$P_{n0}(x) = \int_{R^1} \alpha_{1/n}(x-y)\tilde{P}_0(y - \frac{2}{n})\, dy.$$

When n is large enough such that $P_0(0) - \frac{3A}{n} > 0$ and $n > a + 2$, it is easy to verify that $P_{n0}(x)$ satisfies the hypotheses (A_3), $P_{n0}(x) \to P_0(x)$ uniformly on $\bar{R}^+$ as $n \to \infty$ and $\|P_{n0}'(x)\|_{L^\infty} \leq \|P_0'(x)\|_{L^\infty}$. Using $P_{n0}(x)$ as an initial function, we consider problem II'. It follows from Theorem 4.1 that for each n and ϵ, there exists a $T^*(n,\epsilon)$ such that for $0 < T \leq T^*(n,\epsilon)$, Problem II' has a unique solution P_n^ϵ satisfying the properties (i)-(vii) of Theorem 4.1. Setting $T^* = \inf_{n,\epsilon} T^*(n,\epsilon)$, we assert that $T^* > 0$. In fact, if $T^* = 0$, we can select a decreasing sequence $T^*(n,\epsilon)$ such that $T^*(n_k,\epsilon_k) \to 0$ as $k \to \infty$. On the other hand, by the definition of $T^*(n_k,\epsilon_k)$, we have $P_{n_k}^{\epsilon_k}(0, T^*(n_k,\epsilon_k)) = 0$. On noting

$$\begin{aligned} P_{n_k0}(0) = P_{n_k}^{\epsilon_k}(0,0) &= |P_{n_k}^{\epsilon_k}(0,0) - P_{n_k}^{\epsilon_k}(0, T^*(n_k,\epsilon_k)| \\ &\leq C(T^*(n_k,\epsilon_k))^{1/2}, \end{aligned}$$

and letting $k \to \infty$, this yields that $P_0(0) = 0$, which is contradictory with the condition $P_0(0) > 0$.

For any $T \leq T^*$, using Theorem 4.1 and Ascoli-Arzela's theorem, from sequence $\{P_n^\epsilon(x,t)\}$ we can select a sequence $\{P_{n_j}^{\epsilon_j}(x,t)\}$ such that

$$P_{n_j}^{\epsilon_j}(x,t) \to P(x,t) \text{ uniformly on } [0,N] \times [0,T];$$

$$P_{n_j x}^{\epsilon_j}(x,t) \to P_x(x,t) \text{ uniformly on any compact subset in } (0,N) \times (0,T);$$

$$(P_{n_j x}^{\epsilon_j}(x,t) + \epsilon_j)^m \to (P_x(x,t))^m \text{ weakly in } L^2((0,N) \times (0,T))$$

as $\epsilon_j \to 0$, $n_j \to \infty$, for an arbitrary large integer N and any $0 < T \leq T^*$.

It is easy to verify that the limit function $P(x,t)$ satisfies all the requirements of Definition 1.2 and the properties of Theorem 5.2.

6. Regularity.

We have constructed a solution of Problem I and a solution of Problem II. We state here some regularity properties of these solutions for the two problems. The techniques of the proof are similar to that used in the porous medium equation for gas flow in porous media (cf. Aronson 1969, Gilding and Peletier 1976). We omit the proof and directly state the following results:

Theorem 6.1. *Let $P(x,t)$ be a solution satisfying Definition 1.1, then $P(x,t)$ has the following properties:*

(i) For any $\tau \in (0,T)$, any small $\eta > 0$, there is a constant $C = C(m, P_w, P_e, \|P_0'(x)\|_{L^\infty}, \eta, \tau)$ such that

$$|(P_x)^{m-1}(x,s) - (P_x)^{m-1}(y,t)| \leq C[|x-y| + |s-t|^{1/2}],$$

for $\eta \leq x,\ y < \infty,\ \tau \leq s,\ t < \infty$.

(ii) The derivatives $((P_x)^m)_x$ and P_t exist and are continuous on $R^+ \times (0,\infty)$. Thus, in an ordinary sense $P(x,t)$ satisfies the equation

$$P_t = ((P_x)^m)_x \ \text{ in } R^+ \times (0,\infty).$$

(iii) If $1 < m < 2$ then P_{xx} exists and is continuous in $R^+ \times (0,\infty)$.

Theorem 6.2. *Let $P(x,t)$ be a solution satisfying Definition 1.1, then $P(x,t)$ has the following properties:*

(i) For any $\tau \in (0,T)$, any small $\eta > 0$, there is a constant $C = C(m, A, P_e, \|P_0'(x)\|_{L^\infty}, \eta, \tau)$ such that

$$|(P_x)^{m-1}(x,s) - (P_x)^{m-1}(y,t)| \leq C[|x-y| + |s-t|^{1/2}],$$

for $\eta \leq x,\ y < \infty,\ \tau \leq s,\ t \leq T$.

(ii) The derivatives $((P_x)^m)_x$ and P_t exist and are continuous on $R^+ \times (0,\infty)$. Thus, in an ordinary sense $P(x,t)$ satisfies the equation

$$p_t = (P_x)^m)_x \quad in\ R^+ \times (0,\infty).$$

(iii) If $1 < m < 2$ then P_{xx} exists and is continuous in $R^+ \times (0,\infty)$.

7. Conclusions.

In this chapter, a mathematical analysis has been presented for two typical initial-boundary value problems of unsteady flow of power law fluids in porous media. The problems studied here correspond to well-test cases in the oil industry. The existence of solution is obtained as the limit of a sequence of classical solutions to an approximate non-degenerate equation with the initial-boundary value conditions. The derivation of a Hölder estimate for the approximate solutions is a decisive step in this study. The uniqueness and regularity of the solutions are also discussed.

Chapter 7 Conclusions

In this thesis we have studied five different problems, using some non-Darcian models, in porous media. Some general conclusions, limitations and direction for future work can now be drawn.

In the first problem, a Cartesian-tensor solution of the Brinkman equation was given and applied to two problems. The method is simple and easy to develop. It can be generalized by considering higher order tensors than those considered in this thesis. It may also be generalized to consider the behavior of non-spherical particles. The method has, however, some limitations. In particular, it can only be applied to those problems for which the boundary conditions are either already given in the Cartesian-tensor form or which are expressible in this manner.

The second problem was concerned with the theoretical analysis of a steady convection problem in porous media. A class of weak solutions for relevant partial differential equation was defined and the existence, regularity and uniqueness of weak solutions was discussed. This problem can be generalized by including inertial terms in the Brinkman equations and also by considering more general boundary conditions than those considered in this thesis. The task of solving these problems will, however, not be easy.

The third and fourth problems dealt with the stability of Rayleigh-Bénard convection problems in porous media. The third problem considered the linear stability analysis of the Brinkman equation with anisotropic permeability. The results obtained fitted neatly between the low porosity Darcy approximation and pure viscous fluid. This work can be generalized by considering anisotropy in thermal diffusivity. The fourth problem considered nonlinear stability analysis of a rotating porous layer based on the Brinkman-Boussinesq model. A newly developed energy method was applied to solve this problem and the results obtained match very well with the known results in the limiting cases. The method can be applied to other problems for which the traditional energy method fails to produce useful results. The method, however, requires considerable experience in selecting an appropriate Liapunov function which is the key factor in arriving at the final results.

The last problem employed a power law fluid model to study two typical initial-boundary value problems for a quasilinear degenerate parabolic differential equation. A rigorous theoretical analysis was given concerning the existence, uniqueness and regularity of the solutions of the two problems. Here only problems corresponding to well-test cases were considered. Other problems with different boundary value conditions and initial value conditions can be considered. Moreover, problems associated with the present study, such as the moving boundary problem and determinnig the waiting time at which the stationary interface begins to move, are interesting topics. It should also be pointed out that results on the non-Newtonian flow through porous media are quite scare. This is an interesting area for future work.

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